

Singular Hochschild Cohomology and Gerstenhaber Algebra Structure

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Abstract

In this paper, we define the singular Hochschild cohomology groups $\mathrm{HH}_{\mathrm{sg}}^i(A, A)$ of an associative k -algebra A as morphisms from A to $A[i]$ in the singular category $\mathcal{D}_{\mathrm{sg}}(A \otimes_k A^{\mathrm{op}})$ for $i \in \mathbb{Z}$. We prove that $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ has a Gerstenhaber algebra structure and in the case of a symmetric algebra A , $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ is a Batalin-Vilkovisky (BV) algebra.

1 Introduction

Let A be an associative algebra over a commutative ring k such that A is projective as a k -module. Then the Hochschild cohomology groups $\mathrm{HH}^i(A, A)$ can be defined as morphisms from A to $A[i]$ in the bounded derived category $\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})$ of the enveloping algebra $A \otimes_k A^{\mathrm{op}}$ for $i \in \mathbb{Z}_{\geq 0}$. Namely, we have

$$\mathrm{HH}^i(A, A) := \mathrm{Hom}_{\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})}(A, A[i]).$$

M. Gerstenhaber showed in [Ger] that there is a very rich structure on $\mathrm{HH}^*(A, A)$. More precisely, he proved that $\mathrm{HH}^*(A, A)$ is a so-called Gerstenhaber algebra. Namely, there is a Gerstenhaber bracket $[\cdot, \cdot]$ such that $[\cdot, \cdot]$ is a Lie bracket of degree -1 , and a graded commutative associative cup product \cup , such that $[\cdot, \cdot]$ is a graded derivation of the cup product \cup in each variable.

In this paper, we will generalize the Hochschild cohomology groups to define the singular Hochschild cohomology groups $\mathrm{HH}_{\mathrm{sg}}^i(A, A)$ for $i \in \mathbb{Z}$. Namely, we define the singular Hochschild cohomology groups as

$$\mathrm{HH}_{\mathrm{sg}}^i(A, A) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A \otimes_k A^{\mathrm{op}})}(A, A[i]),$$

where $\mathcal{D}_{\mathrm{sg}}(A \otimes_k A^{\mathrm{op}})$ is the singular category of $A \otimes_k A^{\mathrm{op}}$. Recall that $\mathcal{D}_{\mathrm{sg}}(A \otimes_k A^{\mathrm{op}})$ is the Verdier quotient of $\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})$ by the full subcategory consisting of perfect complexes, that is, bounded complexes of projective A - A -bimodules (cf. [Buch, Or]). We observe that $\mathrm{HH}_{\mathrm{sg}}^*(A, A) = 0$ for an algebra A of finite global dimension since the singular category $\mathcal{D}_{\mathrm{sg}}(A \otimes_k A^{\mathrm{op}})$ is zero in this case. So, from this point of view, the algebras we are interested in are those of infinite global dimension. Note that in general, $\mathrm{HH}_{\mathrm{sg}}^i(A, A)$ does not vanish even for $i \in \mathbb{Z}_{< 0}$. The main result of this paper is as follow.

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Theorem 1.1 (=Theorem 4.1). *Let A be an associative algebra over a commutative ring k such that A is projective as a k -module. Then the singular Hochschild cohomology*

$$\mathrm{HH}_{\mathrm{sg}}^*(A, A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{HH}_{\mathrm{sg}}^i(A, A)$$

is a Gerstenhaber algebra, equipped with a Gerstenhaber bracket $[\cdot, \cdot]$ and the Yoneda product \cup in the singular category $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$.

From Buchweitz's work in his manuscript [Buch], we have a nice description on the singular Hochschild cohomology $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ for a self-injective algebra A . Namely, suppose that A is a self-injective algebra over a field k (e.g. a group algebra of a finite group). Then we have

$$\mathrm{HH}_{\mathrm{sg}}^i(A, A) \cong \begin{cases} \mathrm{HH}^i(A, A) & \text{if } i > 0, \\ \mathrm{HH}_{-i-1}(A, \mathrm{Hom}_{A^e}(A, A^e)) & \text{if } i < -1. \end{cases}$$

Here we remark that, from [CiSo], for the case of a group algebra $k[G]$, where G is a finite abelian group and the characteristic of k divides the order of G , $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ is very related to the Tate cohomology $\widehat{\mathrm{HH}}^*(G, k)$ of G with coefficients in k , the trivial kG -module. We also remark that in the case of a self-injective algebra A , the singular Hochschild cohomology $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ agrees with the Tate-Hochschild cohomology defined in [BeJo] and [BeJoOp] and the stable Hochschild cohomology defined in [EuSc].

Recall that L. Menichi in [Men2] and T. Tradler in [Tra] independently showed that the Hochschild cohomology $\mathrm{HH}^*(A, A)$ of a finite dimensional symmetric algebra A has a new structure, the so-called Batalin-Vilkovisky (BV) structure (c.f. Definition 6.14), which has been studied in topology and mathematical physics during several decades. Roughly speaking a BV structure is a differential operator on Hochschild cohomology and it is a “generator” of the Gerstenhaber bracket $[\cdot, \cdot]$, which means that $[\cdot, \cdot]$ is the obstruction of the differential operator being a graded derivation with respect to the cup product. In this paper, we will generalize this result and prove that the singular Hochschild cohomology $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ of a finite dimensional symmetric algebra A has a BV algebra structure. Namely, we have the following result.

Theorem 1.2 (=Corollary 6.21). *Let A be a symmetric algebra over a field k . Then the singular Hochschild cohomology $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ is a BV algebra with the BV operator Δ_{sg} , which is the Connes B -operator for the negative part $\mathrm{HH}_{\mathrm{sg}}^{\leq 0}(A, A)$, the Δ -operator for the positive part $\mathrm{HH}_{\mathrm{sg}}^{\geq 0}(A, A)$ and*

$$\Delta_{\mathrm{sg}}|_{\mathrm{HH}_{\mathrm{sg}}^0(A, A)} = 0 : \mathrm{HH}_{\mathrm{sg}}^0(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A).$$

In particular, we have two BV subalgebras $\mathrm{HH}_{\mathrm{sg}}^{\leq 0}(A, A)$ and $\mathrm{HH}_{\mathrm{sg}}^{\geq 0}(A, A)$ with induced BV algebra structures.

As a corollary, we obtain that the cyclic homology $\mathrm{HC}_*(A, A)[-1]$ of a symmetric algebra A , has a graded Lie algebra structure (cf. Corollary 6.22). We remark that L. Menichi showed that the negative cyclic cohomology $\mathrm{HC}_-^*(A, A)[-1]$ (cf. Proposition 25, [Men1]) and the cyclic cohomology $\mathrm{HC}_\lambda^*(A)[-1]$ (cf. Corollary 43, [Men2]) are both a graded Lie algebra. So in some sense our result is a dual version of his results.

Throughout this paper, we fix a commutative ring k with unit. We assume that all rings and the modules are simultaneously k -modules and that all operations on rings and modules are naturally k -module homomorphisms. For simplicity, we often use the symbol \otimes to represent \otimes_k , the tensor product over the commutative base ring k . For a k -algebra A , we denote $(a_i \otimes a_{i+1} \otimes \cdots \otimes a_j) \in A^{\otimes j-i+1} (i \leq j)$ sometimes by $a_{i,j}$ for short, and denote the enveloping algebra $A \otimes_k A^{\text{op}}$ by A^e .

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2 Preliminaries

In this section we recall some notions on Hochschild cohomology and Gerstenhaber algebras. For more details, we refer the reader to [Lod, Ger].

Definition 2.1. Let k be a commutative ring with unit. A differential graded Lie algebra (DGLA) is a differential \mathbb{Z} -graded k -module (L, d) with a bracket $[\cdot, \cdot] : L^i \times L^j \rightarrow L^{i+j}$ which satisfies the following properties:

1. it is skew-symmetric:

$$[\alpha, \beta] = -(-1)^{|\alpha||\beta|}[\beta, \alpha];$$

2. satisfies the graded Leibniz rule:

$$d([\alpha, \beta]) = (-1)^{|\beta|}[d\alpha, \beta] + [\alpha, d\beta];$$

3. and the graded Jacobi identity:

$$(-1)^{(|\alpha|-1)(|\gamma|-1)}[[\alpha, \beta], \gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)}[[\beta, \gamma], \alpha] + (-1)^{(|\gamma|-1)(|\beta|-1)}[[\gamma, \alpha], \beta] = 0,$$

where α, β, γ are arbitrary homogeneous elements in (L, d) and $|\alpha|$ is the degree of the homogeneous element α .

Remark 2.2. Let $(L, d, [\cdot, \cdot])$ be a DGLA. Then the homology $H^*(L, d)$ of the differential graded module (L, d) is a \mathbb{Z} -graded Lie algebra with the induced bracket $[\cdot, \cdot]$.

Definition 2.3. A Gerstenhaber algebra is a \mathbb{Z} -graded k -module $\mathcal{H}^* := \bigoplus_{n \in \mathbb{Z}} \mathcal{H}^n$ equipped with:

1. a graded commutative associative product \cup of degree zero, with unit $1 \in \mathcal{H}^0$,

$$\begin{aligned} \cup : \mathcal{H}^m \times \mathcal{H}^n &\rightarrow \mathcal{H}^{m+n} \\ (\alpha, \beta) &\mapsto \alpha \cup \beta. \end{aligned}$$

In particular, $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$;

2. a graded Lie algebra structure $[\cdot, \cdot]$ on $\mathcal{H}^*[-1]$, that is,

$$[\alpha, \beta] = -(-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, \alpha]$$

and

$$(-1)^{(|\alpha|-1)(|\gamma|-1)} [[\alpha, \beta], \gamma] + (-1)^{(|\beta|-1)(|\alpha|-1)} [[\beta, \gamma], \alpha] + (-1)^{(|\gamma|-1)(|\beta|-1)} [[\gamma, \alpha], \beta] = 0;$$

3. compatibility between \cup and $[\cdot, \cdot]$:

$$[\alpha, \beta \cup \gamma] = \beta \cup [\alpha, \gamma] + (-1)^{|\gamma|(|\alpha|-1)} [\alpha, \beta] \cup \gamma,$$

(or equivalently,

$$[\alpha \cup \beta, \gamma] = [\alpha, \gamma] \cup \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \cup [\beta, \gamma])$$

where α, β, γ are arbitrary homogeneous elements in \mathcal{H}^* and $|\alpha|$ is the degree of the homogeneous element α .

We follow [BaGi] to define a Gerstenhaber module of a Gerstenhaber algebra \mathcal{H}^* .

Definition 2.4. A Gerstenhaber module of \mathcal{H}^* is a \mathbb{Z} -graded vector space \mathcal{F}^* equipped with:

1. a module structure \cup' for the graded algebra (\mathcal{H}^*, \cup) ;
2. a module structure $[\cdot, \cdot]'$ for the graded Lie algebra $(\mathcal{H}^*[-1], [\cdot, \cdot])$. That is,

$$[[\alpha, \beta], x]' = [\alpha, [\beta, x]]' - (-1)^{(|\alpha|-1)(|\beta|-1)} [\beta, [\alpha, x]]';$$

3. compatibility:

$$[\alpha \cup \beta, x]' = (-1)^{|\alpha|(|x|-1)} \alpha \cup' [\beta, x]' + (-1)^{(|\alpha|+|x|-1)|\beta|} \beta \cup' [\alpha, x]',$$

$$[\alpha, \beta \cup' x]' = \beta \cup' [\alpha, x]' + (-1)^{|x|(|\alpha|-1)} [\alpha, \beta] \cup' x$$

where α, β are arbitrary homogeneous elements in \mathcal{H}^* and x is arbitrary homogeneous element in \mathcal{F}^* .

Remark 2.5. For any Gerstenhaber algebra \mathcal{H}^* , \mathcal{H}^* is a Gerstenhaber module over itself.

Naturally, we can define morphisms between Gerstenhaber modules.

Definition 2.6. Let $(\mathcal{H}^*, \cup, [\cdot, \cdot])$ be a Gerstenhaber algebra over a commutative ring k . Let $(\mathcal{F}_1^*, \cup_1, [\cdot, \cdot]_1)$ and $(\mathcal{F}_2^*, \cup_2, [\cdot, \cdot]_2)$ be two Gerstenhaber modules of \mathcal{H}^* . We say that a k -module morphism $\varphi : \mathcal{F}_1^* \rightarrow \mathcal{F}_2^*$ is a Gerstenhaber morphism of degree r , ($r \in \mathbb{Z}$) if the following two conditions are satisfied

1. φ is a module morphism of degree r for the graded commutative algebra (\mathcal{H}^*, \cup) . That is, for any $f \in \mathcal{H}^m$ and $g \in \mathcal{F}_1^*$,

$$\varphi(f \cup_1 g) = (-1)^{mr} f \cup_2 \varphi(g)$$

2. φ is a module morphism of degree r for the graded Lie algebra $(\mathcal{H}^*[-1], [\cdot, \cdot])$. That is, for any $f \in \mathcal{H}^m$ and $g \in \mathcal{F}_1^*$,

$$\varphi([f, g]_1) = (-1)^{(m-1)r} [f, \varphi(g)]_2.$$

The classical example of Gerstenhaber algebras is the Hochschild cohomology

$$\mathrm{HH}^*(A, A) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathrm{HH}^n(A, A)$$

of an associative algebra A . Let us start to recall some notions on Hochschild cohomology. Let A be an associative algebra over k such that A is projective as a k -module and M be an A - A -bimodule. Recall that the Hochschild cohomology of A with coefficients in M is defined as

$$\mathrm{HH}^*(A, M) := \mathrm{Ext}_{A^e}^*(A, M),$$

and Hochschild homology is defined as

$$\mathrm{HH}_*(A, M) := \mathrm{Tor}_*^{A^e}(A, M)$$

where $A^e := A \otimes_k A^{\mathrm{op}}$ is the enveloping algebra of A . Recall that we have the following (un-normalized) bar resolution of A ,

$$\mathrm{Bar}_*(A) : \dots \longrightarrow A^{\otimes(r+2)} \xrightarrow{d_r} A^{\otimes(r+1)} \longrightarrow \dots \longrightarrow A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{d_0 := \mu} A, \quad (1)$$

where μ is the multiplication of A and d_r is defined as follows,

$$d_r(a_0 \otimes a_1 \otimes \dots \otimes a_{r+2}) = \sum_{i=0}^r (-1)^i a_{0,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,r+1}.$$

Denote

$$\begin{aligned} C^r(A, M) &:= \mathrm{Hom}_{A^e}(A^{\otimes r+2}, M), \\ C_r(A, M) &:= M \otimes_{A^e} A^{\otimes r+2}, \end{aligned}$$

for any $r \in \mathbb{Z}_{\geq 0}$. Note that

$$\begin{aligned} \mathrm{Hom}_{A^e}(A^{\otimes r+2}, M) &\cong \mathrm{Hom}_k(A^{\otimes r}, M), \\ M \otimes_{A^e} A^{\otimes r+2} &\cong M \otimes_k A^{\otimes r}. \end{aligned}$$

We also consider the the normalized bar resolution $\overline{\mathrm{Bar}}_*(A)$, which is defined as

$$\overline{\mathrm{Bar}}_r(A) := A \otimes \overline{A}^{\otimes r} \otimes A,$$

where $\overline{A} := A/(k \cdot 1_A)$, with induced differential in $\mathrm{Bar}_*(A)$. Thus the Hochschild cohomology $\mathrm{HH}^*(A, M)$ can be computed by the following complex,

$$C^*(A, M) : M \xrightarrow{\delta^0} \mathrm{Hom}_k(A, M) \longrightarrow \dots \longrightarrow \mathrm{Hom}_k(A^{\otimes r}, M) \xrightarrow{\delta^r} \mathrm{Hom}_k(A^{\otimes r+1}, M) \longrightarrow \dots,$$

where δ^r is defined as follows, for any $f \in \text{Hom}_k(A^{\otimes r}, M)$

$$\begin{aligned} \delta^r(f)(a_1 \otimes \cdots \otimes a_{r+1}) : &= a_1 f(a_{2,r+1}) + \sum_{i=1}^r (-1)^i f(a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,r+1}) + \\ &(-1)^{r+1} f(a_{1,r}) a_{r+1}. \end{aligned}$$

We denote

$$Z^r(A, M) := \text{Ker}(\delta^r)$$

and

$$B^r(A, M) := \text{Im}(\delta^{r-1})$$

for any $r \in \mathbb{Z}_{\geq 0}$. Then we have

$$\text{HH}^r(A, M) = Z^r(A, M)/B^r(A, M).$$

We can also compute the Hochschild cohomology $\text{HH}^*(A, A)$ via the normalized Bar resolution $\overline{\text{Bar}}_*(A)$, namely, we have (cf. e.g. [Lod])

$$\text{HH}^r(A, M) = \overline{Z}^r(A, M)/\overline{B}^r(A, M),$$

where $\overline{Z}^r(A, M)$ and $\overline{B}^r(A, M)$ are respectively the r -th cocycle and r -th coboundary in the normalized cochain complex $\overline{C}^*(A, M)$.

The Hochschild homology $\text{HH}_*(A, M)$ is the homology of the following complex,

$$C_*(A, M) : \cdots \longrightarrow C_r(A, M) \xrightarrow{\partial_r} C_{r-1}(A, M) \longrightarrow \cdots \longrightarrow C_0(A, M)$$

where

$$\partial_r(m \otimes a_1 \otimes \cdots \otimes a_r) := m a_1 \otimes a_{2,r} + \sum_{i=1}^{r-1} (-1)^i m \otimes a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,r} + (-1)^r a_r m \otimes a_{1,r-1}.$$

Denote

$$\begin{aligned} B_r(A, M) &:= \text{Im}(\partial_{r+1}) \\ Z_r(A, M) &:= \text{Ker}(\partial_r). \end{aligned}$$

Then we have

$$\text{HH}_r(A, M) := Z_r(A, M)/B_r(A, M).$$

Similarly, it also can be computed by the normalized bar resolution $\overline{\text{Bar}}_*(A, M)$, namely, we have (cf. e.g. [Lod])

$$\text{HH}_r(A, M) = \overline{Z}_r(A, M)/\overline{B}_r(A, M).$$

Let us recall the cup product,

$$\cup : C^m(A, A) \times C^n(A, M) \rightarrow C^{m+n}(A, M),$$

which is defined in the following way. Given $f \in C^m(A, A)$ and $g \in C^n(A, M)$,

$$(f \cup g)(a_1 \otimes \cdots \otimes a_{m+n}) := f(a_{1,m}) g(a_{m+1,m+n}).$$

One can check that this cup product \cup induces a well-defined operation (still denoted by \cup) on cohomology groups, that is,

$$\cup : \mathrm{HH}^m(A, A) \times \mathrm{HH}^n(A, M) \rightarrow \mathrm{HH}^{m+n}(A, M).$$

Recall that there is also a circ product,

$$\circ : C^m(A, M) \times C^n(A, A) \rightarrow C^{m+n-1}(A, M)$$

which is defined as follows, given $f \in C^m(A, A)$ and $g \in C^n(A, M)$, for $1 \leq i \leq n$, set

$$\begin{aligned} g \circ_i f(a_1 \otimes \cdots \otimes a_{m+n-1}) &:= g(a_{1,i-1} \otimes f(a_{i,i+m-1}) \otimes a_{i+m,m+n-1}), \\ g \circ f &:= \sum_{i=1}^n (-1)^{(m-1)(i-1)} g \circ_i f, \end{aligned} \tag{2}$$

for $n = 0$, we set

$$g \circ f := 0.$$

Using this circ product, one can define a Lie bracket $[\cdot, \cdot]$ on $\mathrm{HH}^*(A, A)$ in the following way. Let $f \in C^m(A, A)$ and $g \in C^n(A, A)$, define

$$[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f.$$

One can check that this Lie bracket induces a well-defined Lie bracket (still denoted by $[\cdot, \cdot]$) on $\mathrm{HH}^*(A, A)$.

With these two operators \cup and $[\cdot, \cdot]$ on $\mathrm{HH}^*(A, A)$, Gerstenhaber proves the following result.

Theorem 2.7 ([Ger]). *Let A be an associative algebra over a commutative ring k . Then the Hochschild cohomology $\mathrm{HH}^*(A, A)$, equipped with the cup product \cup and bracket $[\cdot, \cdot]$ is a Gerstenhaber algebra.*

Remark 2.8. Let A be an associative k -algebra such that A is projective as a k -module. Then for any $m \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{HH}^m(A, A) \cong \mathrm{Hom}_{\mathcal{D}^b(A^e)}(A, A[m]),$$

and the cup product \cup can be interpreted as compositions of morphisms (namely, the Yoneda product) in $\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})$.

At the end of this section, let us recall the cap product \cap , which is an action of Hochschild cohomology on Hochschild homology. For any $r, p \in \mathbb{Z}_{\geq 0}$ such that $r \geq p$, there is a bilinear map

$$\cap : C_r(A, M) \otimes C^p(A, A) \rightarrow C_{r-p}(A, M)$$

sending $(m \otimes a_1 \otimes \cdots \otimes a_r) \otimes \alpha$ to

$$(m \otimes a_1 \otimes \cdots \otimes a_r) \cap \alpha := (-1)^{rp} (m \otimes_A \alpha(a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1} \otimes \cdots \otimes a_r)$$

It is straightforward to verify that \cap induces a well-defined map, which we still denote by \cap , on the level of homology,

$$\cap : \mathrm{HH}_r(A, M) \otimes \mathrm{HH}^p(A, A) \rightarrow \mathrm{HH}_{r-p}(A, M). \tag{3}$$

3 Singular Hochschild cohomology

Let A be an associative algebra over a commutative ring k such that A is projective as a k -module. Recall the un-normalized bar resolution $\text{Bar}_*(A)$ (cf. (1)) of A ,

$$\cdots \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A^{\otimes 2} \xrightarrow{d_0} A \longrightarrow 0.$$

Let us denote the p -th kernel $\text{Ker}(d_{p-1})$ in the un-normalized bar resolution $\text{Bar}_*(A)$ by $\Omega^p(A)$. Then we have the following short exact sequence for $p \in \mathbb{Z}_{>0}$,

$$0 \rightarrow \Omega^p(A) \rightarrow A^{\otimes(p+1)} \rightarrow \Omega^{p-1}(A) \rightarrow 0 \quad (4)$$

which induces a long exact sequence

$$\cdots \rightarrow \text{HH}^m(A, A^{\otimes(p+1)}) \rightarrow \text{HH}^m(A, \Omega^{p-1}(A)) \rightarrow \text{HH}^{m+1}(A, \Omega^p(A)) \rightarrow \cdots \quad (5)$$

We denote the connecting morphism in (5) by,

$$\theta_{m,p-1} : \text{HH}^m(A, \Omega^{p-1}(A)) \rightarrow \text{HH}^{m+1}(A, \Omega^p(A))$$

for $m \in \mathbb{Z}_{\geq 0}$. Hence we obtain an inductive system for any fixed $m \in \mathbb{Z}_{\geq 0}$,

$$\text{HH}^m(A, A) \xrightarrow{\theta_{m,0}} \text{HH}^{m+1}(A, \Omega^1(A)) \xrightarrow{\theta_{m,1}} \text{HH}^{m+2}(A, \Omega^2(A)) \longrightarrow \cdots \quad (6)$$

Let

$$\varinjlim_{p \in \mathbb{Z}_{\geq 0}} \text{HH}^{m+p}(A, \Omega^p(A))$$

be the colimit of the inductive system (6) above.

Since A is a k -algebra such that A is projective as a k -module, we have a canonical isomorphism

$$\text{HH}^m(A, M) \cong \text{Hom}_{\mathcal{D}^b(A^e)}(A, M[m])$$

for any A - A -bimodule M and $m \in \mathbb{Z}_{\geq 0}$. Hence we have the following morphism for any $m \in \mathbb{Z}, p \in \mathbb{Z}_{\geq 0}$ such that $m+p > 0$,

$$\Phi_{m,p} : \text{HH}^{m+p}(A, \Omega^p(A)) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^e)}(A, \Omega^p(A)[m+p]) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^e)}(A, A[m]), \quad (7)$$

which are compatible with the inductive system (6) above. So the collection of maps $\Phi_{m,p}$ induces a morphism for any fixed $m \in \mathbb{Z}$,

$$\Phi_m : \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \text{HH}^{m+p}(A, \Omega^p(A)) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^e)}(A, A[m]).$$

Next we will prove that Φ_m is an isomorphism for any $m \in \mathbb{Z}$.

Proposition 3.1. *For any $m \in \mathbb{Z}$, the morphism*

$$\Phi_m : \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \text{HH}^{m+p}(A, \Omega^p(A)) \rightarrow \text{Hom}_{\mathcal{D}_{\text{sg}}(A^e)}(A, A[m])$$

defined above is an isomorphism.

Proof. First, let us recall the following fact (cf. e.g. Proposition 6.7.17 [Zim]):

Fact 3.2. *The following canonical homomorphism is an isomorphism for any $m \in \mathbb{Z}$*

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[m]) \cong \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p}(A), \Omega^p(A)).$$

Now using the fact above, we obtain that Φ_m is surjective. Indeed, assume

$$f \in \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[m]),$$

then from the fact above, there exists $p \in \mathbb{Z}_{\geq 0}$ such that f can be represented by some element

$$f' \in \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p}(A), \Omega^p(A)),$$

hence f is also represented by some element $f'' \in \mathrm{Hom}_{A^e}(\Omega^{m+p}(A), \Omega^p(A))$. It follows that f'' induces a cocycle (See Diagram (8))

$$\alpha := f'' \circ d_{m+p} \in \mathrm{Hom}_{A^e}(A^{\otimes m+p+2}, \Omega^p(A)),$$

hence $\alpha \in \mathrm{HH}^{m+p}(A, \Omega^p(A))$.

$$\begin{array}{ccc} A^{\otimes m+p+3} & & \\ d_{m+p+1} \downarrow & & \\ A^{\otimes m+p+2} & \xrightarrow{\alpha} & \Omega^p(A) \\ d_{m+p} \downarrow & \nearrow f'' & \\ \Omega^{m+p}(A) & & \\ \downarrow & & \\ A^{\otimes m+p+1} & & \end{array} \quad (8)$$

Moreover, we have that $\Phi_m(\alpha) = f$, so Φ_m is surjective. Here we remark that the following morphism between two inductive systems,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{HH}^{m+p}(A, \Omega^p(A)) & \longrightarrow & \mathrm{HH}^{m+p+1}(A, \Omega^{p+1}(A)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p}(A), \Omega^p(A)) & \longrightarrow & \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p+1}(A), \Omega^{p+1}(A)) & \longrightarrow & \cdots \end{array}$$

induces a commutative diagram,

$$\begin{array}{ccc} \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \mathrm{HH}^{m+p}(A, \Omega^p(A)) & \xrightarrow{\Phi_m} & \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[m]) \\ \downarrow & \nearrow \cong & \\ \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p}(A), \Omega^p(A)) & & \end{array}$$

It remains to prove that Φ_m is injective. Assume that there exists

$$\beta \in \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \mathrm{HH}^{m+p}(A, \Omega^p(A))$$

such that $\Phi_m(\beta) = 0$. Then there exists $p \in \mathbb{Z}_{\geq 0}$ such that β is represented by some element $\beta' \in \mathrm{HH}^{m+p}(A, \Omega^p(A))$ and β' is mapped into zero under the morphism

$$\mathrm{HH}^{m+p}(A, \Omega^p(A)) \rightarrow \underline{\mathrm{Hom}}_{A^e}(\Omega^{m+p}(A), \Omega^p(A)).$$

So from this it follows that the cocycle β' induces a morphism $g' : \Omega^{m+p}(A) \rightarrow \Omega^p(A)$ such that g' factors through a projective A - A -bimodule P . The maps are illustrated by the following diagram.

$$\begin{array}{ccc} & A^{\otimes m+p+3} & \\ & \downarrow d_{m+p+1} & \\ A^{\otimes m+p+2} & \xrightarrow{\beta'} & \Omega^p(A) \\ & \downarrow d_{m+p} & \nearrow g' \\ \Omega^{m+p}(A) & \xrightarrow{\sigma} & P \\ & \downarrow & \uparrow \tau \\ & A^{\otimes m+p+1} & \end{array} \quad (9)$$

By functoriality, the A - A -bimodule morphism $\tau : P \rightarrow \Omega^p(A)$ in Diagram (9) induces the following map.

$$\begin{array}{ccc} \tau^* : \mathrm{HH}^{m+p}(A, P) & \rightarrow & \mathrm{HH}^{m+p}(A, \Omega^p(A)) \\ \sigma & \mapsto & \beta' \end{array} \quad (10)$$

Since P is a projective A - A -bimodule, we have the following commutative diagram between two short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^{p+1} & \longrightarrow & A^{p+2} & \longrightarrow & \Omega^p(A) \longrightarrow 0 \\ & & \uparrow & & \uparrow \tau' & & \uparrow \tau \\ 0 & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\cong} & P \longrightarrow 0 \end{array}$$

Hence from the functoriality of long exact sequences induced from short exact sequences, we have the following commutative diagram.

$$\begin{array}{ccccccc} \dots & \longrightarrow & \mathrm{HH}^{m+p}(A, A^{\otimes p+2}) & \longrightarrow & \mathrm{HH}^{m+p}(A, \Omega^p(A)) & \xrightarrow{\theta_{m+p,p}} & \mathrm{HH}^{m+p+1}(A, \Omega^{p+1}(A)) \\ & & \uparrow \tau'^* & & \uparrow \tau^* & & \uparrow \\ \dots & \longrightarrow & \mathrm{HH}^{m+p}(A, P) & \xrightarrow{\cong} & \mathrm{HH}^{m+p}(A, P) & \longrightarrow & 0 \end{array} \quad (11)$$

So from Diagram (11) above, we obtain that

$$\theta_{m+p,p}(\beta') = \theta_{m+p,p}\tau^*(\sigma) = 0,$$

thus it follows that β is represented by zero element in $\mathrm{HH}^{m+p+1}(A, \Omega^{p+1}(A))$, hence

$$\beta = 0 \in \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \mathrm{HH}^{m+p}(A, \Omega^p(A)).$$

So Φ_m is injective. Therefore, Φ_m is an isomorphism. ■

Remark 3.3. By the same argument, we also have the following isomorphism for any $m \in \mathbb{Z}$,

$$\overline{\Phi}_m : \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \mathrm{HH}^{m+p}(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[m])$$

where $\overline{\Omega}^p(A)$ is the p -th kernel in the normalized bar resolution of A .

4 Gerstenhaber algebra structure on singular Hochschild cohomology

In this section, we will prove the following main theorem.

Theorem 4.1. *Let A be an associative algebra over a commutative ring k and suppose that A is projective as a k -module. Then the singular Hochschild cohomology (shifted by [1])*

$$\mathrm{HH}_{\mathrm{sg}}^*(A, A)[1] := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[n])[1]$$

is a Gerstenhaber algebra.

For $p \in \mathbb{Z}_{\geq 0}$, we denote the p -th kernel $\mathrm{Ker}(\overline{d}_{p-1})$ in the normalized bar resolution $\overline{\mathrm{Bar}}_*(A)$ by $\overline{\Omega}^p(A)$. Note that $\overline{\Omega}^0(A) = A$.

Let $m, n \in \mathbb{Z}_{>0}$ and $p, q \in \mathbb{Z}_{\geq 0}$. we shall define a Gerstenhaber bracket as follows

$$[\cdot, \cdot] : C^m(A, \overline{\Omega}^p(A)) \otimes C^n(A, \overline{\Omega}^q(A)) \rightarrow C^{m+n-1}(A, \overline{\Omega}^{p+q}(A)). \quad (12)$$

Let $f \in C^m(A, \overline{\Omega}^p(A))$ and $g \in C^n(A, \overline{\Omega}^q(A))$, define

$$f \bullet_i g := \begin{cases} d((f \otimes \mathrm{id}^{\otimes q})(\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{\otimes m-i}) \otimes 1) & \text{if } 1 \leq i \leq m, \\ d((\mathrm{id}^{\otimes -i} \otimes f \otimes \mathrm{id}^{\otimes q+i})(g \otimes \mathrm{id}^{\otimes m-1}) \otimes 1) & \text{if } -q \leq i \leq -1, \end{cases}$$

and

$$f \bullet g := \sum_{i=1}^m (-1)^{r(m,p;n,q;i)} f \bullet_i g + \sum_{i=1}^q (-1)^{s(m,p;n,q;i)} f \bullet_{-i} g$$

where $r(m, p; n, q; i)$ and $s(m, p; n, q; i)$ are defined in the following way,

$$\begin{aligned} r(m, p; n, q; i) &= p + q + (i-1)(q-n-1), \quad 1 \leq i \leq m, \\ s(m, p; n, q; i) &= p + q + i(p-m-1), \quad 1 \leq i \leq q. \end{aligned} \quad (13)$$

Then we define

$$[f, g] := f \bullet g - (-1)^{(m-p-1)(n-q-1)} g \bullet f. \quad (14)$$

Remark 4.2. We have a double complex $C^{*,*}(A, A)$, which is defined by

$$C^{m,p}(A, A) := \begin{cases} C^m(A, \overline{\Omega}^p(A)) & \text{if } m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise} \end{cases}$$

with the horizontal differential $\delta : C^m(A, \overline{\Omega}^p(A)) \rightarrow C^{m+1}(A, \overline{\Omega}^p(A))$, induced from bar resolution and the vertical differential zero. Recall that the total complex of $C^{*,*}(A, A)$, denoted by $\text{Tot}(C^{*,*}(A, A))^*$, is defined as follows,

$$\text{Tot}(C^{*,*}(A, A))^n := \bigoplus_{n=m-p} C^{m,p}(A, A)$$

with the differential induced from $C^{*,*}(A, A)$.

Note that the bullet product \bullet is not associative, however it has the following “weak associativity”.

Lemma 4.3. *Let $f_i \in C^{m_i}(A, \overline{\Omega}^{p_i}(A))$ for $i = 1, 2, 3$.*

1. *For $1 \leq j \leq m_1$ and $1 \leq i \leq m_1 + m_2 - 1$, we have*

$$(f_1 \bullet_j f_2) \bullet_i f_3 = \begin{cases} (-1)^{p_1+p_3} f_1 \bullet_j (f_2 \bullet_{i-j+1} f_3) & \text{if } 0 \leq i-j < m_2, \\ (-1)^{p_3+p_2} (f_1 \bullet_{i+p_2-m_2+1} f_3) \bullet_j f_2 & \text{if } m_2 \leq i-j, i < m_1 + m_2 - p_2 \\ f_3 \bullet_{-(p_1+p_2+1+i-m_1-m_2)} (f_1 \bullet_j f_2) & \text{if } m_2 \leq i-j, m_1 + m_2 - p_2 \leq i \\ (-1)^{p_1+p_3} f_1 \bullet_i (f_2 \bullet_{-(j-i)} f_3) & \text{if } 1 \leq j-i \leq p_3 \\ (-1)^{p_2+p_3} (f_1 \bullet_i f_3) \bullet_{m_3+j-p_3-1} f_2 & \text{if } p_3 < j-i \end{cases}$$

2. *For $1 \leq j \leq p_2$ and $1 \leq i \leq m_1 + m_2 - 1$,*

$$(f_1 \bullet_{-j} f_2) \bullet_i f_3 = \begin{cases} (-1)^{p_1+p_3} f_1 \bullet_{-j} (f_2 \bullet_i f_3) & \text{if } 1 \leq i \leq m_2 \\ (-1)^{p_2+p_3} (f_1 \bullet_{p_2-j+i-m_2+1} f_3) \bullet_{-j} f_2 & \text{if } m_2 < i < m_1 + m_2 - p_2 + j \\ f_3 \bullet_{-(p_1+p_2+1+i-m_1-m_2)} (f_1 \bullet_{-j} f_2) & \text{if } m_2 < m_1 + m_2 - p_2 + j \leq i \\ f_3 \bullet_{-(p_1+p_2+1+i-m_1-m_2)} (f_1 \bullet_{-j} f_2) & \text{if } m_2 < i, m_1 + j \leq p_2 \end{cases}$$

3. *For $1 \leq j \leq m_1$ and $1 \leq i \leq p_3$,*

$$(f_1 \bullet_j f_2) \bullet_{-i} f_3 = \begin{cases} (-1)^{p_1+p_3} f_1 \bullet_{-i} (f_2 \bullet_{-(i+j-1)} f_3) & \text{if } i+j < p_3 - 2 \\ (-1)^{p_2+p_3} (f_1 \bullet_{-i} f_3) \bullet_{m_3-p_3+j+i-1} f_2 & \text{if } p_3 - 2 \leq i+j \end{cases}$$

4. *For $1 \leq j \leq p_2$ and $1 \leq i \leq p_3$,*

$$(f_1 \bullet_{-j} f_2) \bullet_{-i} f_3 = (-1)^{p_1+p_3} f_1 \bullet_{-(i+j)} (f_2 \bullet_{-i} f_3).$$

Similarly, we have the following lemma.

Lemma 4.4. 1. *For $1 \leq j \leq m_1$ and $1 \leq i \leq m_2$,*

$$f_1 \bullet_j (f_2 \bullet_i f_3) = (f_1 \bullet_j f_2) \bullet_{j+i-1} f_3$$

2. *For $1 \leq j \leq m_1$ and $1 \leq i \leq p_3$,*

$$f_1 \bullet_j (f_2 \bullet_{-i} f_3) = \begin{cases} (-1)^{p_1+p_3} (f_1 \bullet_{i+j} f_2) \bullet_j f_3 & \text{if } i+j \leq m_1 \\ (-1)^{p_1+p_2} f_2 \bullet_{-(i+j-m_1+p_1)} (f_1 \bullet_j f_3) & \text{if } m_1 < i+j \end{cases}$$

3. For $1 \leq j \leq p_2 + p_3$ and $1 \leq i \leq m_2$,

$$f_1 \bullet_{-j} (f_2 \bullet_i f_3) = \begin{cases} (-1)^{p_1+p_3} (f_1 \bullet_{-j} f_2) \bullet_i f_3 & \text{if } 0 \leq j \leq p_2 \\ (-1)^{p_1+p_2} f_2 \bullet_i (f_1 \bullet_{-(j+m_2-p_2-i)} f_3) & \text{if } 0 < j - p_2 \leq p_3 + i - m_2 \\ (f_2 \bullet_i f_3) \bullet_{m_2+m_3-p_2-p_3-1+j} f_1 & \text{if } 1 \leq p_3 + i - m_2 + 1 \leq j - p_2 \\ (f_2 \bullet_i f_3) \bullet_{m_2+m_3-p_2-p_3-1+j} f_1 & \text{if } 1 \leq j - p_2, p_3 + i - m_2 < 0 \end{cases}$$

4. For $1 \leq j \leq p_2 + p_3$ and $1 \leq i \leq p_3$,

$$f_1 \bullet_{-j} (f_2 \bullet_{-i} f_3) = \begin{cases} (-1)^{p_1+p_3} (f_1 \bullet_{i-j+1} f_2) \bullet_{-j} f_3 & \text{if } 0 \leq i - j \\ (-1)^{p_1+p_3} (f_1 \bullet_{-(j-i)} f_2) \bullet_{-i} f_3 & \text{if } 0 < j - i \leq p_2 \\ (f_2 \bullet_{-i} f_3) \bullet_{m_2+m_3-p_2-p_3-1+j} f_1 & \text{if } p_2 < j - i \end{cases}$$

Remark 4.5. Similar to [Ger], the cup product \cup for $C^{*,*}(A, A)$ can be expressed by the multiplication μ and the bullet product \bullet . Namely, for $f \in C^{m,p}(A, A)$ and $g \in C^{n,q}(A, A)$,

$$f \cup g = (-1)^q (\mu \bullet_{-p} f) \bullet_{m+1} g.$$

Indeed, we have

$$\begin{aligned} (\mu \bullet_{-p} f) \bullet_{m+1} g(a_{1,m+n}) &= d(\mu \bullet_{-p} f \otimes \text{id})(a_{1,m} \otimes g(a_{m+1,m+n})) \otimes 1 \\ &= (-1)^p d(\text{id}_p \otimes \mu)(f(a_{1,m}) \otimes g(a_{m+1,m+n})) \otimes 1 \\ &= (-1)^q f(a_{1,m}) g(a_{m+1,m+n}). \end{aligned}$$

The differential δ in $C^{*,*}(A, A)$ can also be expressed using the multiplication and the bullet product \bullet . Namely, let $f \in C^{m,p}(A, A)$, then

$$\delta(f) = [f, -\mu] = (-1)^{m-p-1} [\mu, f].$$

Indeed, by definition, we have,

$$\begin{aligned} [f, -\mu](a_{1,m+1}) &= \sum_{i=1}^m (-1)^{i+p} f \bullet_i \mu(a_{1,m+1}) + \sum_{i=1}^2 (-1)^{i(m-p-1)+p} \mu \bullet_i f(a_{1,m+1}) + \\ &\quad \sum_{i=1}^p (-1)^{i+m-1} \mu \bullet_{-i} f \\ &= \sum_{i=1}^m (-1)^i f(a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,m+1}) + (-1)^{m-1} d(\mu \otimes \text{id})(f(a_{1,m}) \otimes a_{m+1}) \otimes 1 + \\ &\quad a_1 f(a_{2,m+1}) + \sum_{i=1}^p (-1)^{i+m-1} d(\text{id}_i \otimes \mu)(f(a_{1,m}) \otimes a_{m+1}) \otimes 1 \\ &= \sum_{i=1}^m (-1)^i f(a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,m+1}) + a_1 f(a_{2,m+1}) + (-1)^m f(a_{1,m}) a_{m+1} \\ &= \delta(f)(a_{1,m}) \end{aligned}$$

where we used the fact that $d(\overline{\Omega}^p(A)) = 0$ and $d \circ d = 0$ in the third identity. Note that the bullet product \bullet does not define a preLie algebra structure (defined in [Ger]) on $\text{Tot}(C^*(A, A))$, in general. However, in the following proposition we will show that the bullet product \bullet defines a (graded) Lie-admissible algebra structure (defined in Section 2.2 of [MeVa]) on $\text{Tot}(C^*(A, A))$. That is, the associated Lie bracket $[\cdot, \cdot]$ defines a differential graded Lie algebra structure on $\text{Tot}(C^*(A, A))$.

Proposition 4.6. *The bracket $[\cdot, \cdot]$ (defined in (14)) gives a differential graded Lie algebra (DGLA) structure on the total complex (shifted by $[1]$)*

$$\text{Tot}(C^{*,*}(A, A))^*[1].$$

As a consequence,

$$\text{HH}^{>0}(A, \overline{\Omega}^*(A))[1] := \bigoplus_{\substack{m \in \mathbb{Z}_{>0}, \\ p \in \mathbb{Z}_{\geq 0}}} \text{HH}^m(A, \overline{\Omega}^p(A))[1]$$

is a \mathbb{Z} -graded Lie algebra, with the grading

$$\text{HH}^{>0}(A, \overline{\Omega}^*(A))_n := \bigoplus_{\substack{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0} \\ m-p=n}} \text{HH}^m(A, \overline{\Omega}^p(A)).$$

Proof. From the definition of the bracket in (14), we observe that $[\cdot, \cdot]$ is skew symmetric,

$$[f_1, f_2] = -(-1)^{(m_1-p_1-1)(m_2-p_2-1)}[f_2, f_1].$$

Now let us check the Jacobi identity. That is, let $f_i \in C^{m_i}(A, \overline{\Omega}^{p_i}(A))$ for $i = 1, 2, 3$, we need to check that

$$(-1)^{n_1 n_3} [[f_1, f_2], f_3] + (-1)^{n_2 n_1} [[f_2, f_3], f_1] + (-1)^{n_3 n_2} [[f_3, f_1], f_2] = 0 \quad (15)$$

where $n_i := m_i - p_i - 1$, for $i = 1, 2, 3$. Now let us apply Lemma 4.3 and 4.4, from these two lemmas, we note that every term on the left hand side of Identity (15) will appear exactly twice. So the only thing that we should do is to compare the coefficients of the same two terms. However, this can be done case by case. Let us first consider the coefficient of the term $(f_1 \bullet_j f_2) \bullet_i f_3$ in (15), which is

$$(-1)^{n_1 n_3 + r(m_1, p_1; m_2, p_2; j) + r(m_1 + m_2 - 1, p_1 + p_2; m_3, p_3; i)} = (-1)^{n_1 n_3 + (j-1)n_2 + (i-1)n_3 + p_3}.$$

By Lemma 4.3, we have the following cases

1. If $0 \leq i - j < m_2$, we have

$$(f_1 \bullet_j f_2) \bullet_i f_3 = (-1)^{p_1 + p_3} f_1 \bullet_j (f_2 \bullet_{i-j+1} f_3). \quad (16)$$

The coefficient of the term $f_1 \bullet_j (f_2 \bullet_{i-j+1} f_3)$ is

$$(-1)^{n_1 n_3 + 1 + r(m_2, p_2; m_3, p_3; i-j+1) + r(m_1, p_1; m_2 + m_3 - 1, p_2 + p_3; j)} = (-1)^{n_1 n_3 + 1 + (i-j)n_3 + (j-1)(n_2 + n_3) + p_1}.$$

Hence the coefficients of these two terms in Identity (15) are up to the scale $-(-1)^{p_1 + p_3}$, then from (16), it follows that these two terms will be cancelled in (15).

2. If $m_2 \leq i - j, i < m_1 + m_2 - p_2$, then

$$(f_1 \bullet_j f_2) \bullet_i f_3 = (-1)^{p_3 + p_2} (f_1 \bullet_{i+p_2-m_2+1} f_3) \bullet_j f_2.$$

The coefficient of the term $(f_1 \bullet_{i+p_2-m_2+1} f_3) \bullet_j f_2$ is

$$(-1)^{n_2 n_3 + 1 + n_1 n_3 + r(m_1, p_1; m_3, p_3; i-n_2) + r(m_1 + m_3 - 1, p_1 + p_3; m_2, p_2; j)} = (-1)^{n_1 n_3 + 1 + (j-1)n_2 + (i-1)n_3 + p_2}.$$

Hence the coefficients of these two terms are up to the scale $-(-1)^{p_2 + p_3}$, so they will be cancelled in Identity (15).

3. If $m_2 \leq i - j, m_1 + m_2 - p_2 \leq i$, then

$$(f_1 \bullet_j f_2) \bullet_i f_3 = f_3 \bullet_{-(p_1+p_2+1+i-m_1-m_2)} (f_1 \bullet_j f_2).$$

The coefficient of the term $f_3 \bullet_{-(p_1+p_2+1+i-m_1-m_2)} (f_1 \bullet_j f_2)$ in (15) is

$$(-1)^{n_2 n_3 + 1 + r(m_1, p_1; m_2, p_2; j) + s(m_3, p_3; m_1 + m_2 - 1, p_1 + p_2; n_1 + n_2 - 1 + i)} = (-1)^{n_1 n_3 + 1 + (j-1)n_2 + (i-1)n_3 + p_3}.$$

Hence the coefficients are up to the scale -1 , so in the same reason they will be cancelled.

4. If $1 \leq j - i \leq p_3$, then

$$(f_1 \bullet_j f_2) \bullet_i f_3 = (-1)^{p_1 + p_3} f_1 \bullet_i (f_2 \bullet_{-(j-i)} f_3).$$

The coefficient of the term $f_1 \bullet_i (f_2 \bullet_{-(j-i)} f_3)$ in (15) is

$$(-1)^{n_1 n_2 + 1 + n_1(n_2 + n_3) + r(m_1, p_1; m_2 + m_3 - 1, p_2 + p_3; i) + s(m_2, p_2; m_3, p_3; j - i)} = (-1)^{n_1 n_3 + 1 + n_2(i-1) + n_3(j-1) + p_1}.$$

Hence the coefficients are up to the scale $-(-1)^{p_1 + p_3}$, so they will be cancelled.

5. If $p_3 < j - i$, then

$$(f_1 \bullet_j f_2) \bullet_i f_3 = (-1)^{p_2 + p_3} (f_1 \bullet_i f_3) \bullet_{m_3 + j - p_3 - 1} f_2$$

The coefficient of the term $(f_1 \bullet_i f_3) \bullet_{m_3 + j - p_3 - 1} f_2$ is

$$(-1)^{n_2 n_3 + 1 + n_1 n_3 + r(m_1, p_1; m_3, p_3; i) + r(m_1 + m_3 - 1, p_1 + p_3; m_2, p_2; n_3 + j)} = (-1)^{1 + n_1 n_3 + n_3(i-1) + (j-1)n_2 + p_2}.$$

Hence the coefficients are up to the scale $-(-1)^{p_2 + p_3}$, so they will be cancelled.

In a similar way, the other cases can be checked. So Identity (15) holds. Hence, it remains to verify the following identity, for $f \in C^{m,p}(A, A)$ and $g \in C^{n,q}(A, A)$,

$$\delta([f, g]) = (-1)^{n-q-1} [\delta(f), g] + [f, \delta(g)]. \quad (17)$$

Now by Remark 4.5, it is equivalent to verify that

$$[[f, g], \mu] = (-1)^{n-q-1} [[f, \mu], g] + [f, [g, \mu]],$$

which is exactly followed from the Jacobi identity. Hence Identity (17) holds. Therefore, $\text{Tot}(C^{*,*}(A, A))^*[1]$ is a differential graded Lie algebra. Observe that for any $n \in \mathbb{Z}$, we have,

$$H^n(\text{Tot}(C^{*,*}(A, A))^*) \cong \bigoplus_{\substack{m \in \mathbb{Z}_{>0}, p \in \mathbb{Z}_{\geq 0} \\ m-p=n}} \text{HH}^m(A, \overline{\Omega}^p(A))$$

hence

$$\text{HH}^{>0}(A, \overline{\Omega}^*(A))[1] := \bigoplus_{\substack{m \in \mathbb{Z}_{>0}, \\ p \in \mathbb{Z}_{\geq 0}}} \text{HH}^m(A, \overline{\Omega}^p(A))[1]$$

is a \mathbb{Z} -graded Lie algebra since the homology of a differential graded Lie algebra is a graded Lie algebra with the induced Lie bracket. \blacksquare

Let $m \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$. Then we have a short exact sequence (from the normalized bar resolution of A),

$$0 \longrightarrow \overline{\Omega}^{p+1}(A) \xrightarrow{\iota_{p+1}} A^{\otimes(p+2)} \xrightarrow{d_p} \overline{\Omega}^p(A) \longrightarrow 0,$$

which induces a long exact sequence,

$$\cdots \longrightarrow \mathrm{HH}^m(A, \overline{\Omega}^p(A)) \xrightarrow{\theta_{m,p}} \mathrm{HH}^{m+1}(A, \overline{\Omega}^{p+1}(A)) \longrightarrow \mathrm{HH}^{m+1}(A, A^{\otimes(p+2)}) \longrightarrow \cdots \quad (18)$$

where

$$\theta_{m,p} : \mathrm{HH}^m(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{HH}^{m+1}(A, \overline{\Omega}^{p+1}(A))$$

is the connecting homomorphism. In fact, we can write $\theta_{m,p}$ explicitly. For any $f \in C^m(A, \overline{\Omega}^p(A))$,

$$\theta_{m,p}(f)(a_{1,m+1}) = (-1)^p d(f(a_{1,m}) \otimes a_{m+1} \otimes 1). \quad (19)$$

Indeed, $\theta_{m,p}$ is induced from the following lifting,

$$\begin{array}{ccccc} A \otimes \overline{A}^{\otimes m} \otimes A & \xleftarrow{d} & A \otimes \overline{A}^{\otimes m+1} \otimes A & & \\ \downarrow \overline{f} & \searrow f_0 & & \searrow f_1 & \\ \overline{\Omega}^p(A) & \xleftarrow{d} & A \otimes \overline{A}^{\otimes p} \otimes A & \xleftarrow{d} & A \otimes \overline{A}^{\otimes p+1} \otimes A \end{array}$$

where

$$\begin{aligned} \overline{f}(a_{1,m+2}) &= a_1 f(a_{2,m+1}) a_{m+2}, \\ f_0(a_{1,m+2}) &= (-1)^p a_1 f(a_{2,m+1}) \otimes a_{m+2}, \\ f_1(a_{1,m+3}) &= (-1)^m a_1 f(a_{2,m+1}) \otimes a_{m+1} \otimes a_{m+2}. \end{aligned}$$

Hence $\theta_{m,p}(f)(a_{1,m+1}) = (-1)^p d(f(a_{1,m}) \otimes a_{m+1} \otimes 1)$. As a result, these connecting homomorphisms $\theta_{m,p}$ induce a homomorphism of degree zero between \mathbb{Z} -graded vector spaces,

$$\theta : \mathrm{HH}^*(A, \overline{\Omega}^*(A)) \rightarrow \mathrm{HH}^*(A, \overline{\Omega}^*(A)) \quad (20)$$

where

$$\theta|_{\mathrm{HH}^m(A, \overline{\Omega}^p(A))} = \theta_{m,p} : \mathrm{HH}^m(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{HH}^{m+1}(A, \overline{\Omega}^{p+1}(A)).$$

The following proposition shows that θ is a module homomorphism of the \mathbb{Z} -graded Lie algebra $\mathrm{HH}^*(A, \overline{\Omega}^*(A))$.

Proposition 4.7. *Let A be an associative algebra over a commutative algebra k , then the homomorphism of \mathbb{Z} -graded k -modules*

$$\theta : \mathrm{HH}^*(A, \overline{\Omega}^*(A)) \rightarrow \mathrm{HH}^*(A, \overline{\Omega}^*(A))$$

(defined in (20) above) is a module homomorphism of degree zero over the \mathbb{Z} -graded Lie algebra $\mathrm{HH}^(A, \overline{\Omega}^*(A))$.*

Proof. Let $f \in \mathrm{HH}^m(A, \overline{\Omega}^p(A))$ and $g \in \mathrm{HH}^n(A, \overline{\Omega}^q(A))$, it is sufficient to verify that

$$\theta_{m+n-1,p+q}([f, g]) = [\theta_{m,p}(f), g]. \quad (21)$$

First we claim that the following two identities hold,

$$\begin{aligned}\theta_{m,p}(f) \bullet g - \theta_{m+n-1,p+q}(f \bullet g) &= (-1)^{m(q-n-1)+p+q} d(f \otimes g \otimes 1) \\ g \bullet \theta_{m,p}(f) - \theta_{m+n-1,p+q}(g \bullet f) &= (-1)^{(p+1)(q-n-1)+p+q} d(f \otimes g \otimes 1).\end{aligned}\tag{22}$$

It is easy to check that (22) implies Identity (21). Now let us verify the two identities in (22). Indeed, we have

$$\begin{aligned}\theta_{m,p}(f) \bullet g &= \sum_{i=1}^{m+1} (-1)^{r(m+1,p+1;n,q;i)} \theta_{m,p}(f) \bullet_i g + \sum_{i=1}^q (-1)^{s(m+1,p+1;n,q;i)} \theta_{m,p}(f) \bullet_{-i} g \\ &= \sum_{i=1}^m (-1)^{r(m,p;n,q;i)} \theta_{m+n-1,p+q}(f \bullet_i g) + \sum_{i=1}^q (-1)^{s(m,p;n,q;i)} \theta_{m,p}(f) \bullet_{-i} g + \\ &\quad (-1)^{r(m+1,p+1;n,q;m+1)} \theta_{m,p}(f) \bullet_{m+1} g \\ &= \theta_{m+n-1,p+q}(f \bullet g) + (-1)^{m(q-n-1)+p+q} d(f \otimes g \otimes 1)\end{aligned}$$

Similarly, we have

$$\begin{aligned}g \bullet \theta_{m,p}(f) &= \sum_{i=1}^n (-1)^{r(n,q;m+1,p+1;i)} g \bullet_i \theta_{m,p}(f) + \sum_{i=1}^{p+1} (-1)^{s(n,q;m+1,p+1;i)} g \bullet_{-i} \theta_{m,p}(f) \\ &= \sum_{i=1}^n (-1)^{r(n,q;m,p;i)} \theta_{m+n-1,p+1}(g \bullet_i f) + \sum_{i=1}^p (-1)^{s(n,q;m,p;i)} \theta_{m+n-1,p+1}(g \bullet_{-i} f) + \\ &\quad (-1)^{s(n,q;m+1,p+1;p+1)} g \bullet_{-p-1} \theta_{m,p}(f) \\ &= \theta_{m+n-1,p+q}(g \bullet f) + (-1)^{(p+1)(q-n-1)+p+q} d(f \otimes g \otimes 1).\end{aligned}$$

Hence we have proved that the two identities in (22) hold, so the proof is completed. \blacksquare

Remark 4.8. Note that we did not consider $\mathrm{HH}^0(A, \Omega^p(A))$ for $p \in \mathbb{Z}_{>0}$ when we defined the Lie bracket $[\cdot, \cdot]$ on $\mathrm{HH}^{>0}(A, \overline{\Omega}^*(A))$. In fact, for $p \in \mathbb{Z}_{\geq 0}$, let us consider the homomorphism

$$\theta_{0,p} : \mathrm{HH}^0(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{HH}^1(A, \overline{\Omega}^{p+1}(A)).$$

Via this homomorphism, we can define a Lie bracket action of $\mathrm{HH}^0(A, \Omega^p(A))$ on $\mathrm{HH}^n(A, \Omega^q(A))$ for $n \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$ as follows: for $\alpha \in \mathrm{HH}^0(A, \Omega^p(A))$ and $g \in \mathrm{HH}^n(A, \Omega^q(A))$, define

$$[\alpha, g] := [\theta_{0,p}(\alpha), g].$$

Denote

$$\mathrm{HH}^{\geq 0}(A, \overline{\Omega}^*(A)) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}, p \in \mathbb{Z}_{\geq 0}} \mathrm{HH}^m(A, \overline{\Omega}^p(A)).$$

In the following proposition, we will prove that there is a Gerstenhaber algebra structure on $\mathrm{HH}^{\geq 0}(A, \overline{\Omega}^*(A))$.

Proposition 4.9. *Let A be an associative algebra over a commutative ring k . Suppose that A is projective as a k -module. Then*

$$\mathrm{HH}^{\geq 0}(A, \overline{\Omega}^*(A)) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}, p \in \mathbb{Z}_{\geq 0}} \mathrm{HH}^m(A, \overline{\Omega}^p(A))$$

with the Lie bracket $[\cdot, \cdot]$ and the cup product \cup , is a Gerstenhaber algebra.

Proof. From Proposition 4.6, Proposition 4.7 and Remark 4.8, it follows that $\mathrm{HH}^{\geq 0}(A, \overline{\Omega}^*(A))$ is a \mathbb{Z} -graded Lie algebra. Next let us prove that \cup defines a graded commutative algebra structure. Let $f_i \in \mathrm{HH}^{m_i}(A, \overline{\Omega}^{p_i}(A))$ for $i = 1, 2$. Then we claim that

$$f_1 \cup f_2 - (-1)^{(m_1-p_1)(m_2-p_2)} f_2 \cup f_1 = \delta(f_2 \bullet f_1),$$

indeed, by immediate calculation, we have

$$\begin{aligned} \delta(f_2 \bullet f_1)(a_{1,m+n}) &= \sum_{i=1}^{m_2} (-1)^{p_1+p_2+(i-1)(p_1-m_1-1)} \delta(f_2 \circ_i f_1)(a_{1,m+n}) + \\ &\quad \sum_{i=1}^{p_1} (-1)^{p_1+p_2+i(p_2-m_2-1)} \delta(f_2 \circ_{-i} f_1)(a_{1,m+n}) \\ &= f_1(a_{1,m_1}) f_2(a_{m_1+1, m_1+m_2}) - (-1)^{(m_1-p_1)(m_2-p_2)} f_2(a_{1, m_2}) f_1(a_{m_2+1, m_1+m_2}). \end{aligned}$$

Hence we obtain, in $\mathrm{HH}^{m_1+m_2}(A, \overline{\Omega}^{p_1+p_2}(A))$

$$f_1 \cup f_2 = (-1)^{(m_1-p_1)(m_2-p_2)} f_2 \cup f_1.$$

So the graded commutativity of cup products holds. It remains to verify the compatibility between \cup and $[\cdot, \cdot]$, namely, for $f_i \in \mathrm{HH}^{m_i}(A, \overline{\Omega}^{p_i}(A))$,

$$[f_1 \cup f_2, f_3] = [f_1, f_3] \cup f_2 + (-1)^{(m_3-p_3-1)(m_1-p_1)} f_1 \cup [f_2, f_3]. \quad (23)$$

Claim 4.10. *In the cohomology group $\mathrm{HH}^{m_1+m_2+m_3-1}(A, \overline{\Omega}^{p_1+p_2+p_3}(A))$, we also have the following identity,*

$$f_3 \bullet (f_1 \cup f_2) - (-1)^{(m_2-p_2)(m_3-p_3-1)} (f_3 \bullet f_1) \cup f_2 - f_1 \cup (f_3 \bullet f_2) = 0. \quad (24)$$

Claim 4.11. *In the cohomology group $\mathrm{HH}^{m_1+m_2+m_3-1}(A, \overline{\Omega}^{p_1+p_2+p_3}(A))$, we have the following identity,*

$$(f_1 \cup f_2) \bullet f_3 - (f_1 \bullet f_3) \cup f_2 - (-1)^{(m_1-p_1)(m_3-p_3-1)} f_1 \cup (f_2 \bullet f_3) = 0. \quad (25)$$

It is easy to check that these two claims imply Identity (23). Now let us prove these two claims. The proofs of these two claims are very similar to the proof of Theorem 5 in [Ger].

Proof of Claim 4.10. First, it is easy to check that we have the following identity for the left hand side in (24)

$$\begin{aligned} \text{LHS} &= \sum_{i=1}^{m_3} (-1)^{r(m_3, p_3; m_1+m_2, p_1+p_2; i)} f_3 \bullet_i (f_1 \cup f_2) + \sum_{i=1}^{p_1} (-1)^{s(m_3, p_3; m_1+m_2, p_1+p_2; i)} f_3 \bullet_{-i} (f_1 \cup f_2) - \\ &\quad \sum_{i=1}^{m_3} (-1)^{(m_2-p_2)(m_3-p_3-1)+r(m_3, p_3; m_1, p_1; i)} (f_3 \bullet_i f_1) \cup f_2 - \\ &\quad \sum_{i=1}^{p_1} (-1)^{(m_2-p_2)(m_3-p_3-1)+s(m_3, p_3; m_1, p_1; i)} (f_3 \bullet_{-i} f_1) \cup f_2 - \sum_{i=1}^{m_3} (-1)^{r(m_3, p_3; m_2, p_2; i)} f_1 \cup (f_3 \bullet_i f_2). \end{aligned}$$

Set

$$H_1 := \sum_{i=1}^{p_1} \sum_{j=1}^{m_3-1} (-1)^{\epsilon_{i,j}} d(\text{id}^{\otimes i} \otimes f_3 \otimes \text{id}^{\otimes p_1+p_2-i})(f_1 \otimes \text{id}^{\otimes j-1} \otimes f_2 \otimes \text{id}^{\otimes m_3-j-1}) \otimes 1 + \\ \sum_{i=1}^{m_3} \sum_{j=1}^{m_2-1} (-1)^{\epsilon'_{i,j}} d((f_3 \otimes \text{id}^{\otimes p_1+p_2})(\text{id}^{\otimes i-1} \otimes f_1 \otimes \text{id}^{\otimes j-1} \otimes f_2 \otimes \text{id}^{\otimes m_3-i-j}) \otimes 1).$$

where to simplify the formula, we do not consider the sign of each term in H_1 . Then we will show that

$$\text{LHS} = \delta(H_1).$$

Namely, take any element $a_{1,m_1+m_2+m_3-1} \in A^{\otimes m_1+m_2+m_3-1}$, we need to prove that

$$\text{LHS}(a_{1,m_1+m_2+m_3-1}) = \delta(H_1)(a_{1,m_1+m_2+m_3-1}). \quad (26)$$

Indeed, this identity can be proved by a recursive procedure. That is, as a first step, we will verify that those terms containing $f_1(a_{1,m_1})$ in (26) can be cancelled by frequently using the fact that $d^2 = 0$ and $d(\overline{\Omega}^p(A)) = 0$ for $p \in \mathbb{Z}_{>0}$. This can be done by comparing the terms containing $f_1(a_{1,m_1})$ of both sides in (26). Then after cancelling those terms containing $f_1(a_{1,m_1})$, we obtain a new identity

$$\text{LHS}'(a_{1,m_1+m_2+m_3-1}) = \delta(H_1)'(a_{1,m_1+m_2+m_3-1}),$$

which does not have the terms containing $f_1(a_{1,m_1})$. By similar processing, we will cancel the terms containing $f_1(a_{2,m_1+1})$. Hence after several times, all the terms will be cancelled, so Identity (26) holds. Therefore $\text{LHS} = 0$ in the cohomology group $\text{HH}^{m_1+m_2+m_3-1}(A, \overline{\Omega}^{p_1+p_2+p_3}(A))$. \blacksquare

Proof of Claim 4.11, Similarly, the left hand side in (25) can be written as

$$\text{LHS} = \sum_{i=1}^{m_1} (-1)^{r(m_1+m_2,p_1+p_2;m_3,p_3;i)} (f_1 \cup f_2) \bullet_i f_3 + \sum_{i=1}^{p_3} (-1)^{s(m_1+m_2,p_1+p_2;m_3,p_3;i)} (f_1 \cup f_2) \bullet_{-i} f_3 - \\ \sum_{i=1}^{m_1} (-1)^{r(m_1,p_1;m_3,p_3;i)} (f_1 \bullet_i f_3) \cup f_2 - \sum_{i=1}^{p_3} (-1)^{s(m_1,p_1;m_3,p_3;i)} (f_1 \bullet_{-i} f_3) \cup f_2 - \\ \sum_{i=1}^{p_3} (-1)^{(m_1-p_1)(m_3-p_3-1)+s(m_2,p_2;m_3,p_3;i)} f_1 \cup (f_2 \bullet_{-i} f_3).$$

Set

$$H_2 := \sum_{i=1}^{m_1} \sum_{j=1}^{p_3} (-1)^{\epsilon_{i,j}} d((f_1 \otimes \text{id}^{\otimes j-1} \otimes f_2 \otimes \text{id}^{\otimes p_3-j})(\text{id}^{\otimes i-1} \otimes f_3 \otimes \text{id}^{\otimes m_1+m_2-i-1}) \otimes 1) + \\ \sum_{i=1}^{p_3} (-1)^{\epsilon_i} d((\text{id} \otimes f_1 \otimes \text{id}^{\otimes i-1} \otimes f_2 \otimes \text{id}^{\otimes p_3-i-1})(f_3(a_{1,m_3}) \otimes a_{m_3,m_1+m_2+m_3-2}) \otimes 1)$$

Similarly as above, by calculation, we obtain that

$$\text{LHS} = \delta(H_2),$$

hence $\text{LHS} = 0$ in the cohomology group $\text{HH}^{m_1+m_2+m_3-1}(A, \overline{\Omega}^{p_1+p_2+p_3}(A))$. \blacksquare
Therefore, we have completed the proof. \blacksquare

Remark 4.12. The proof of Proposition 4.9 above relies on combinatorial calculations. To understand the Gerstenhaber algebra structure much better, it is interesting to investigate whether there is a B_∞ -algebra structure on the total complex of $C^{*,*}(A, A)$ since from Section 5.2 in [GeJo] it follows that a B_∞ -algebra structure on a chain complex C induces a canonical Gerstenhaber algebra structure on the homology $H_*(C)$.

Now let us prove the main theorem (cf. Theorem 4.1) in this section.

Proof of Theorem 4.1. From Proposition 3.1, we have the following isomorphism for $m \in \mathbb{Z}$,

$$\Phi_m : \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ m+p > 0}} \mathrm{HH}^{m+p}(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A^e)}(A, A[m]).$$

From Proposition 4.7 and Proposition 4.9, it follows that the structural morphism in the direct system $\mathrm{HH}^{m+*}(A, \overline{\Omega}^{p+*}(A))$,

$$\theta_{m,p} : \mathrm{HH}^m(A, \overline{\Omega}^p(A)) \rightarrow \mathrm{HH}^{m+1}(A, \overline{\Omega}^{p+1}(A))$$

preserves the Gerstenhaber algebra structure. Therefore, there is an induced Gerstenhaber algebra structure on its direct limit

$$\mathrm{HH}_{\mathrm{sg}}^*(A, A) \cong \varinjlim_{\substack{p \in \mathbb{Z}_{\geq 0} \\ *+p > 0}} \mathrm{HH}^{*+p}(A, \overline{\Omega}^p(A)).$$

So $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ is a Gerstenhaber algebra, equipped with the cup product \cup and the induced Lie bracket $[\cdot, \cdot]$. ■

Let us denote, for $m \in \mathbb{Z}_{\geq 0}$,

$$\mathrm{Ker}^{m,\infty}(A, A) := \ker(\mathrm{HH}^m(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^m(A, A)),$$

and

$$\mathrm{HH}^{m,\infty}(A, A) := \mathrm{Im}(\mathrm{HH}^m(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^m(A, A)).$$

Then we have

$$\mathrm{HH}^{m,\infty}(A, A) = \mathrm{HH}^m(A, A) / \mathrm{Ker}^{m,\infty}(A, A).$$

Corollary 4.13. $\mathrm{Ker}^{*,\infty}(A, A)$ is a Gerstenhaber ideal of $\mathrm{HH}^*(A, A)$. In particular, $\mathrm{HH}^{*,\infty}(A, A)$ is a Gerstenhaber subalgebra of $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$.

Proof. First let us claim that the natural morphism

$$\Phi : \mathrm{HH}^*(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A)$$

is a Gerstenhaber algebra homomorphism. In order to prove this claim, from Proposition 4.7 and Remark 4.8, it is sufficient to verify that

$$[\theta_{0,0}(\mathrm{HH}^0(A, A)), \theta_{0,0}(\mathrm{HH}^0(A, A))] = 0,$$

where we recall that

$$\theta_{0,0} : \mathrm{HH}^0(A, A) \rightarrow \mathrm{HH}^1(A, \overline{\Omega}^1(A))$$

is the connecting homomorphism in (18). Let $\lambda, \mu \in \mathrm{HH}^0(A, A)$, then from (19) it follows that for any $a \in A$,

$$\begin{aligned} \theta_{0,0}(\lambda)(a) &= \lambda \otimes a - \lambda a \otimes 1; \\ \theta_{0,0}(\mu)(a) &= \mu \otimes a - \mu a \otimes 1. \end{aligned}$$

Hence, by direct calculation, we obtain that for any $a \in A$,

$$[\theta_{0,0}(\lambda), \theta_{0,0}(\mu)](a) = 0.$$

So we have shown that

$$\Phi : \mathrm{HH}^*(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^*(A, A)$$

is a Gerstenhaber algebra homomorphism. Thus its kernel $\mathrm{Ker}^{*,\infty}(A, A)$ is a Gerstenhaber ideal and its image $\mathrm{HH}^{*\infty}(A, A)$ is a Gerstenhaber algebra. \blacksquare

5 Gerstenhaber algebra structure on $\mathrm{HH}^*(A, A^{\otimes >0})$

Let $m, n \in \mathbb{Z}_{>0}$ and $p, q \in \mathbb{Z}_{>1}$, we will define a star product as follows,

$$\star : C^m(A, A^{\otimes p}) \times C^n(A, A^{\otimes q}) \rightarrow C^{m+n-1}(A, A^{\otimes p+q-1})$$

for $f \in C^m(A, A^{\otimes p})$ and $g \in C^n(A, A^{\otimes q})$, denote

$$\begin{aligned} f \star_0 g &:= (f \otimes \mathrm{id})(\mathrm{id}_{m-1} \otimes g), \\ f \star_1 g &:= (\mathrm{id}_{p-1} \otimes f)(g \otimes \mathrm{id}). \end{aligned}$$

Then we define

$$f \star g := (-1)^{(m-1)(n-1)} f \star_0 g + f \star_1 g$$

and denote

$$\{f, g\} := f \star g - (-1)^{(m-1)(n-1)} g \star f. \quad (27)$$

Remark 5.1. For the case $p = 1$, we can also define a Lie bracket

$$\{\cdot, \cdot\} : C^m(A, A) \times C^n(A, A^{\otimes q}) \rightarrow C^{m+n-1}(A, A^{\otimes q}), \quad (28)$$

let $f \in C^m(A, A)$ and $g \in C^n(A, A^{\otimes q})$,

$$\{f, g\} = f \star g - (-1)^{(m-1)(n-1)} g \circ f,$$

where $g \circ f$ is the circ product (cf. (2)) defined in [Ger].

We remark that in general, the star product \star is not associative. However, some associativity properties hold. In the following lemma, we list some of them.

Lemma 5.2. *Let $f_i \in C^{m_i}(A, A^{\otimes p_i})$, $i = 1, 2, 3$. Then we have the following,*

1. for $p_1 \in \mathbb{Z}_{>0}, p_2, p_3 \in \mathbb{Z}_{>1}$,

$$\begin{aligned} (f_1 \star_0 f_2) \star_0 f_3 &= f_1 \star_0 (f_2 \star_0 f_3), \\ (f_1 \star_0 f_2) \star_1 f_3 &= (f_1 \star_1 f_3) \star_0 f_2, \\ (f_1 \star_1 f_2) \star_1 f_3 &= f_1 \star_1 (f_2 \star_1 f_3), \end{aligned}$$

2. for $p_1, p_2 \in \mathbb{Z}_{>0}, p_3 \in \mathbb{Z}_{>1}$,

$$f_1 \star_1 (f_2 \star_0 f_3) = f_2 \star_0 (f_1 \star_1 f_3).$$

Note that $C^{*>0}(A, A^{\otimes*>0})$ is a double complex with horizontal and vertical differentials induced from the bar resolution. We consider the total complex

$$\text{Tot}(C^{*>0}(A, A^{\otimes*>0}))^*,$$

which is a differential \mathbb{Z} -graded k -module, with the grading

$$\text{Tot}(C^{*>0}(A, A^{\otimes*>1}))^m = \bigoplus_{p \in \mathbb{Z}_{>0}} C^m(A, A^{\otimes p})$$

and the differential induced by the horizontal differential.

Remark 5.3. We remark that the horizontal differential δ in $C^{*>0}(A, A^{\otimes*>0})$ can be expressed by the star product, circ product and the multiplication μ of A . That is, for $f \in C^m(A, A^{\otimes p})$,

$$\delta(f) = (-1)^{m-1} \{\mu, f\}.$$

Indeed, we have,

$$\begin{aligned} \{\mu, f\}(a_{1,m+1}) &= \mu \star f(a_{1,m+1}) - (-1)^{m-1} f \circ \mu(a_{1,m+1}) \\ &= (-1)^{m-1} (\mu \otimes \text{id})(a_1 \otimes f(a_{2,m+1})) + (\text{id}_{p-1} \otimes \mu)(f(a_{1,m}) \otimes a_{m+1}) - \\ &\quad (-1)^{m-1} \sum_{i=1}^m (-1)^{i-1} f(a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,m+1}) \\ &= (-1)^{m-1} \delta(f)(a_{1,m+1}). \end{aligned}$$

Proposition 5.4. $\text{Tot}(C^{*>0}(A, A^{\otimes*>0}))^*$ is a differential graded Lie algebra (DGLA) with the bracket $\{\cdot, \cdot\}$ defined in (27). As a consequence,

$$\text{HH}^{*>0}(A, A^{\otimes*>0}) = \bigoplus_{m,p \in \mathbb{Z}_{>0}} \text{HH}^m(A, A^{\otimes p})$$

is a \mathbb{Z} -graded Lie algebra with the grading

$$\text{HH}^{*>0}(A, A^{\otimes*>0})^m := \bigoplus_{p \in \mathbb{Z}_{>0}} \text{HH}^m(A, A^{\otimes p}).$$

Proof. Let us prove first that it is a \mathbb{Z} -graded Lie algebra. Note that the bracket is skew-symmetric. We need to verify the Jacobi identity, namely, for $f_i \in C^{m_i}(A, A^{\otimes p_i})$, $i = 1, 2, 3$,

$$(-1)^{n_1 n_3} \{\{f_1, f_2\}, f_3\} + (-1)^{n_2 n_1} \{\{f_2, f_3\}, f_1\} + (-1)^{n_3 n_2} \{\{f_3, f_1\}, f_2\} = 0 \quad (29)$$

where $n_i := m_i - 1$. Let us discuss the following three cases.

Case 1: $p_i \in \mathbb{Z}_{>1}$. From Lemma 5.2, it follows that each term of the right hand side in (29) will appear exactly twice, hence it is sufficient to compare the coefficients of those two same terms. For example, the coefficient of $(f_1 \star_0 f_2) \star_0 f_3$ in (29) is $(-1)^{n_2(n_1+n_3)}$. From Lemma 5.2, we obtain that

$$(f_1 \star_0 f_2) \star_0 f_3 = f_1 \star_0 (f_2 \star_0 f_3).$$

And the coefficient of $f_1 \star_0 (f_2 \star_0 f_3)$ in (29) is $-(-1)^{n_2(n_1+n_3)}$, hence these two same terms will be cancelled in (29). Similarly, we can verify the other terms.

Case 2: $p_1, p_2 = 1$ and $p_3 \in \mathbb{Z}_{>1}$. From Theorem 2 in [Ger], we have the following identity,

$$f_3 \circ \{f_1, f_2\} = (f_3 \circ f_1) \circ f_2 - (-1)^{(m_1-1)(m_2-1)}(f_3 \circ f_2) \circ f_1.$$

Thus, to verify the Jacobi identity (29) is equivalent to verify the following one,

$$\begin{aligned} \{f_1, f_2\} \star f_3 &= f_1 \star (f_2 \star f_3) - (-1)^{(m_2-1)m_3} f_1 \star (f_3 \circ f_2) - (-1)^{(m_1-1)(m_2+m_3-1)}(f_2 \star f_3) \circ f_1 - \\ &\quad (-1)^{(m_1-1)(m_2-1)} f_2 \star (f_1 \star f_3) + (-1)^{(m_1-1)m_2} f_2 \star (f_3 \circ f_1) + \\ &\quad (-1)^{(m_3-1)(m_2-1)}(f_1 \star f_3) \circ f_2. \end{aligned} \quad (30)$$

It is easy, by definition, to check that the following identity holds,

$$\begin{aligned} &(f_1 \circ f_2) \star f_3 + (-1)^{(m_2-1)m_3} f_1 \star (f_3 \circ f_2) - (-1)^{(m_3-1)(m_2-1)}(f_1 \star f_3) \circ f_2 \\ &= (-1)^{(m_1+m_2)(m_3-1)+(m_1-1)(m_2-1)} f_1 \star_0 (f_2 \star_0 f_3) + f_1 \star_1 (f_2 \star_1 f_3). \end{aligned} \quad (31)$$

Similarly, we also have

$$\begin{aligned} &(f_2 \circ f_1) \star f_3 + (-1)^{(m_1-1)m_3} f_2 \star (f_3 \circ f_1) - (-1)^{(m_3-1)(m_1-1)}(f_2 \star f_3) \circ f_1 \\ &= (-1)^{(m_1+m_2)(m_3-1)+(m_1-1)(m_2-1)} f_2 \star_0 (f_1 \star_0 f_3) + f_2 \star_1 (f_1 \star_1 f_3). \end{aligned} \quad (32)$$

We also note that

$$\begin{aligned} &(f_1 \star (f_2 \star f_3) - (-1)^{(m_1-1)(m_2-1)} f_2 \star (f_1 \star f_3)) \\ &= (-1)^{(m_1-1)(m_2+m_3)} f_1 \star_0 (f_2 \star_0 f_3) + f_1 \star_1 (f_2 \star_1 f_3) - \\ &\quad (-1)^{(m_1-1)(m_2+m_3)+(m_2-1)(m_1+m_3)} f_2 \star_0 (f_1 \star_0 f_3) - (-1)^{(m_1-1)(m_2-1)} f_2 \star_1 (f_1 \star_1 f_3). \end{aligned} \quad (33)$$

So combining (31), (32) and (33), we get Identity (30). Hence the Jacobi identity holds.

Case 3: $p_1 = 1, p_2, p_3 \in \mathbb{Z}_{>1}$. First, similar to Case 2, we have the following identities for $\{i, j\} = \{2, 3\}$.

$$\begin{aligned} &(f_i \circ f_1) \star f_j + (-1)^{(m_1-1)m_j} f_2 \star (f_3 \circ f_1) - (-1)^{(m_1-1)(m_j-1)}(f_i \star f_j) \circ f_1 \\ &= (-1)^{(m_1+m_i)(m_j-1)+(m_1-1)(m_i-1)} f_2 \star_0 (f_1 \star_0 f_j) - f_i \star_1 (f_1 \star_1 f_j). \end{aligned} \quad (34)$$

Hence, each term in Jacobi identity in (29) can be expressed by star product (no circ product). So, similar to Case 1, by using Lemma 5.2, we can compare the same two terms. Thus, we have verified that $\text{Tot}(C^{*>0}(A, A^{\otimes*>0}))^*$ is a \mathbb{Z} -graded Lie algebra.

Next we will prove that $\{\cdot, \cdot\}$ is compatible with differential, that is, let $f_i \in C^{m_i}(A, A^{\otimes p_i})$ for $i = 1, 2$, we need to verify the following identity,

$$\delta(\{f_1, f_2\}) = (-1)^{m_1-1} \{f_1, \delta(f_2)\} + \{\delta(f_1), f_2\}. \quad (35)$$

From Remark 5.3, we note that it follows from the Jacobi identity (29). Since

$$H^m(\text{Tot}(C^{*>0}(A, A^{\otimes*>0}))) \cong \text{HH}^{*>0}(A, A^{\otimes*>0})^m,$$

it follows that $\text{HH}^{*>0}(A, A^{\otimes*>0})$ is a \mathbb{Z} -graded Lie algebra. Hence we have completed the proof. \blacksquare

Lemma 5.5. *Let $f_i \in \text{HH}^{m_i}(A, A^{\otimes p_i})$ for $i = 1, 2$ and suppose that $p_2 > 1$. Then we have $f_1 \cup f_2 = f_2 \cup f_1 = 0$.*

Proof. In fact, we have the following identity for $f_i \in C^{m_i}(A, A^{\otimes p_i})$ and $p_2 > 1$,

$$\begin{aligned}\delta(f_1 \star_0 f_2) &= \delta(f_1) \star_0 f_2 + (-1)^{m_1-1} f_1 \star_0 \delta(f_2) + (-1)^{m_1} f_1 \cup f_2, \\ \delta(f_1 \star_1 f_2) &= f_1 \star_1 \delta(f_2) + (-1)^{m_2-1} \delta(f_1) \star_1 f_2 + (-1)^{m_2} f_2 \cup f_1.\end{aligned}$$

Hence it follows that $f_1 \cup f_2 = f_2 \cup f_1 = 0$. ■

Next we will prove that $\mathrm{HH}^{*>0}(A, A^{\otimes * > 0})$ is a Gerstenhaber algebra.

Proposition 5.6. $\mathrm{HH}^{*>0}(A, A^{\otimes * > 0})$ is a Gerstenhaber algebra (without the unity) with the cup product and Lie bracket.

Proof. It remains to verify the compatibility between cup product and Lie bracket. On the other hand, from Lemma 5.5 above, it is sufficient to verify that for $f_i \in \mathrm{HH}^{m_i}(A, A)$, $i = 1, 2$ and $f_3 \in \mathrm{HH}^{m_3}(A, A^{\otimes p_3})$, where $p_3 > 1$,

$$\{f_1 \cup f_2, f_3\} = (-1)^{m_1(m_3-1)} f_1 \cup \{f_2, f_3\} + (-1)^{(m_1+m_3-1)m_2} f_2 \cup \{f_1, f_3\} \quad (36)$$

Recall that we have the following identity in $\mathrm{HH}^{m_1+m_2+n-1}(A, A^{\otimes p})$ (cf. Theorem 5. [Ger]),

$$f_3 \circ (f_1 \cup f_2) - f_1 \cup (f_3 \circ f_2) - (-1)^{m_2(m_3-1)} (f_3 \circ f_1) \cup f_2 = 0. \quad (37)$$

Hence it follows that Identity (36) is equivalent to the following identity,

$$(f_1 \cup f_2) \star f_3 - (f_1 \star f_3) \cup f_2 - (-1)^{m_1(m_3-1)} f_1 \cup (f_2 \star f_3) = 0. \quad (38)$$

Let us compute the left hand side in (38),

$$\begin{aligned}& ((f_1 \cup f_2) \star f_3 - (f_1 \star f_3) \cup f_2 - (-1)^{m_1(n-1)} f_1 \cup (f_2 \star f_3)) \\ &= (-1)^{(m_1+m_2-1)(m_3-1)} f_1 \cup (f_2 \star_0 f_3) + (f_1 \star_1 f_3) \cup f_2 - (-1)^{(m_1-1)(m_3-1)} (f_1 \star_0 f_3) \cup f_2 - \\ & \quad (f_1 \star_1 f_3) \cup f_2 - (-1)^{m_1(m_3-1)+(m_2-1)(m_3-1)} f_1 \cup (f_2 \star_0 f_3) - (-1)^{m_1(m_3-1)} f_1 \cup (f_2 \star_1 f_3) \\ &= -(-1)^{(m_1-1)(m_3-1)} (f_1 \star_0 f_3) \cup f_2 - (-1)^{m_1(m_3-1)} f_1 \cup (f_2 \star_1 f_3)\end{aligned} \quad (39)$$

Set

$$G := (-1)^{m_1 m_3} (f_1 \star_0 f_3) \star_1 f_2.$$

By calculation, we obtain that

$$\delta(G) = (-1)^{(m_1-1)(m_3-1)} (f_1 \star_0 f_3) \cup f_2 + (-1)^{m_1(m_3-1)} f_1 \cup (f_2 \star_1 f_3),$$

hence it follows that in $\mathrm{HH}^{m_1+m_2+n-1}(A, A^{\otimes p})$,

$$(f_1 \cup f_2) \star g - (f_1 \star g) \cup f_2 - (-1)^{m_1(n-1)} f_1 \cup (f_2 \star g) = 0.$$

Therefore, we have verified Identity (36). ■

Corollary 5.7. Let A be an associative algebra over a commutative ring k . Let $f_i \in \mathrm{HH}^{m_i}(A, A)$, $m_i \in \mathbb{Z}_{>0}$ for $i = 1, 2$. Then for any $g \in \mathrm{HH}^m(A, A^{\otimes p})$, $p \in \mathbb{Z}_{>1}$, $m > 0$, we have

$$\{f_1 \cup f_2, g\} = 0.$$

Proof. This result follows from Lemma 5.5 and Proposition 5.6. ■

6 Special case: self-injective algebra

In this section, let A be a self-injective algebra over a field k . Then $A^e := A \otimes_k A^{\text{op}}$ is also a self-injective algebra. Recall that the singular category $\mathcal{D}_{\text{sg}}(A)$ of a self-injective algebra A is equivalent to the stable module category $A\text{-}\underline{\text{mod}}$ as triangulated categories, namely, we have the following proposition.

Proposition 6.1 ([KeVo, Ric]). *Let A be a self-injective algebra. Then the following natural functor is an equivalence of triangulated categories,*

$$A\text{-}\underline{\text{mod}} \rightarrow \mathcal{D}_{\text{sg}}(A).$$

Recall that for any $m \in \mathbb{Z}$, we define

$$\text{HH}_{\text{sg}}^m(A, A) := \text{Hom}_{\mathcal{D}_{\text{sg}}(A^e)}(A, A[m]).$$

Thanks to Corollary 6.4.1 in [Buch], we have the following descriptions for $\text{HH}_{\text{sg}}^*(A, A)$ in the case of a self-injective algebra A .

Proposition 6.2 (Corollary 6.4.1, [Buch]). *Let A be a self-injective algebra over a field k , denote $A^\vee := \text{Hom}_{A^e}(A, A^e)$. Then*

1. $\text{HH}_{\text{sg}}^i(A, A) \cong \text{HH}^i(A, A)$ for all $i > 0$,
2. $\text{HH}_{\text{sg}}^{-i}(A, A) \cong \text{Tor}_{i-1}^{A^e}(A, A^\vee)$ for all $i \geq 2$,
3. *there is an exact sequence*

$$0 \longrightarrow \text{HH}_{\text{sg}}^{-1}(A, A) \longrightarrow A^\vee \otimes_{A^e} A \longrightarrow \text{Hom}_{A^e}(A, A) \longrightarrow \text{HH}_{\text{sg}}^0(A, A) \rightarrow 0.$$

Remark 6.3. Since A is self-injective, so is A^e . Hence we have for $i \geq 2$ and $n \geq 1$,

$$\text{Tor}_{i-1}^{A^e}(A, A^\vee) \cong \text{HH}_{\text{sg}}^{-i}(A, A) \cong \underline{\text{Hom}}_{A^e}(\Omega^i(A), \Omega^{n+i}(A)) \cong \text{Ext}_{A^e}^n(A, \Omega^{n+i}(A)).$$

To simplify the notation, we denote each of these isomorphisms by $\lambda_{i,n}$. In fact, we can write the isomorphism $\lambda_{i,n}$ explicitly, for

$$\alpha \otimes a_1 \otimes \cdots \otimes a_{i-1} \in \text{Tor}_{i-1}^{A^e}(A, A^\vee),$$

we have

$$\lambda_{i,n}(\alpha \otimes a_1 \otimes \cdots \otimes a_{i-1}) \in \text{Ext}_{A^e}^n(A, \Omega^{n+i}(A))$$

which is defined as follows,

$$\lambda_{i,n}(\alpha \otimes a_1 \otimes \cdots \otimes a_{i-1})(b_1 \otimes \cdots \otimes b_n) = \sum_j d(x_j \otimes a_{1,i-1} \otimes y_j \otimes b_{1,n} \otimes 1), \quad (40)$$

where we write $\alpha(1) := \sum_j x_j \otimes y_j$.

Under the isomorphisms $\lambda_{i,n}$ above, we have the following relations between the cap product and cup product.

Lemma 6.4. *Let A be a self-injective algebra. Then we have the following commutative diagram for $m \geq 1, n \geq 2$ and $n - m \geq 2$.*

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^{m-n}(A, A) \\ \downarrow \mathrm{id} \otimes \lambda_n & & \downarrow \lambda_{n-m} \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee) & \xrightarrow{\cap} & \mathrm{Tor}_{n-m-1}^{A^e}(A, A^\vee). \end{array}$$

Proof. Take $f \otimes (\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1}) \in \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee)$. Then

$$f \cap (\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1}) = \sum_i x_i f(a_{1,m}) \otimes y_i \otimes a_{m+1,n-1} \in \mathrm{Tor}_{n-m-1}^{A^e}(A, A^\vee),$$

where $\alpha(1) := \sum_i x_i \otimes y_i$. Hence we have

$$\lambda_{n-m,m+1}(f \cap (\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1})) \in \mathrm{Ext}^{m+1}(A, \Omega^{n+1}(A)).$$

From the formula in (40), it follows that for any $(b_1 \otimes \cdots \otimes b_{m+1}) \in A^{\otimes(m+1)}$,

$$\lambda_{n-m,m+1}(f \cap (\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1}))(b_1 \otimes \cdots \otimes b_{m+1}) = \sum_i d(x_i f(a_{1,m}) \otimes a_{m+1,n-1} \otimes y_i \otimes b_{1,m+1} \otimes 1).$$

We consider the following diagram,

$$\begin{array}{ccccc} \mathrm{Ext}_{A^e}^m(A, A) \otimes \mathrm{Ext}_{A^e}^1(A, \Omega^{n+1}(A)) & \xrightarrow{\cup} & \mathrm{Ext}_{A^e}^{m+1}(A, \Omega^{n+1}(A)) & & \\ \uparrow \lambda_{-m} \otimes \lambda_{n,1} & & \uparrow \lambda_{n-m,m+1} & & \\ \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^{m-n}(A, A) & & \\ \downarrow \mathrm{id} \otimes \lambda_n & & \downarrow \lambda_{n-m} & & \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee) & \xrightarrow{\cap} & \mathrm{Tor}_{n-m-1}^{A^e}(A, A^\vee) & & \end{array} \quad (41)$$

$\mathrm{id} \otimes \lambda_{n,1}$ (left curved arrow from middle-left to top-left)
 $\lambda_{n-m,m+1}$ (right curved arrow from middle-right to top-right)

where \cup in the first row represents the Yoneda product in the bounded derived category $\mathcal{D}^b(A \otimes A^{\mathrm{op}})$. It is clear that the top square in (41) is commutative. Let us prove the commutativity of the outer square. For

$$f \otimes (\alpha \otimes a_1 \otimes \cdots \otimes a_{n-1}) \in \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee),$$

we have (via the up-right direction in Diagram (41))

$$(\cup \circ (\mathrm{id} \otimes \lambda_{n,1}))(f \otimes \alpha \otimes a_{1,n-1}) \in \mathrm{Ext}_{A^e}^{m+1}(A, \Omega^{n+1}(A))$$

which sends $(b_1 \otimes \cdots \otimes b_{m+1}) \in A^{\otimes(m+1)}$ to

$$(\cup \circ (\mathrm{id} \otimes \lambda_{n,1}))(f \otimes \alpha \otimes a_{1,n-1})(b_{1,m+1}) = \sum_i f(b_{1,m}) d(x_i \otimes a_{1,n-1} \otimes y_i \otimes b_{m+1} \otimes 1).$$

On the other hand, we have (via the right-up direction in Diagram (41))

$$(\lambda_{n-m,m+1} \circ \cap)(f \otimes \alpha \otimes a_{1,n-1}) \in \mathrm{Ext}_{A^e}^{m+1}(A, \Omega^{n+1}(A))$$

which sends $(b_1 \otimes \cdots \otimes b_{m+1}) \in A^{\otimes(m+1)}$ to

$$(\lambda_{n-m,m+1} \circ \cap)(f \otimes \alpha \otimes a_{1,n-1})(b_{1,m+1}) = \sum_i d(x_i f(a_{1,m}) \otimes a_{m+1,n-1} \otimes y_i \otimes b_{1,m+1} \otimes 1).$$

For $j = 1, 2, \dots, n$, define

$$H_j \in \text{Hom}_k(A^{\otimes m}, \Omega^{n+1}(A))$$

as follows,

$$H_j(b_{1,m}) := \sum_i d(\text{id}_j \otimes f)(x_i \otimes a_{1,n-1} \otimes y_i \otimes b_{1,m} \otimes 1).$$

By calculation, we have

$$(\cup \circ (\text{id} \otimes \lambda_{n,1}) - \lambda_{n-m,m+1} \circ \cap)(f \otimes \alpha \otimes a_{1,n-1}) = \sum_{j=1}^n (-1)^{\epsilon_j} \delta(H_j),$$

where $\epsilon_j \in \mathbb{Z}$ depends on H_j . Hence in $\text{Ext}_{A^e}^{m+1}(A, \Omega^{n+1}(A))$, we have

$$(\cup \circ (\text{id} \otimes \lambda_{n,1}))(f \otimes \alpha \otimes a_{1,n-1}) = (\lambda_{n-m,m+1} \circ \cap)(f \otimes \alpha \otimes a_{1,n-1}).$$

So we have verified that the outer square in Diagram (41) commutes, hence the lower square also commutes. \blacksquare

Remark 6.5. For the case $n - m = 1$, we have the following commutative diagram

$$\begin{array}{ccc} \text{HH}^m(A, A) \otimes \text{HH}_{\text{sg}}^{-n}(A, A) & \xrightarrow{\cup} & \text{HH}_{\text{sg}}^1(A, A) \\ \downarrow \text{id} \otimes \lambda_n & & \downarrow \lambda_1 \\ \text{HH}^m(A, A) \otimes \text{Tor}_{n-1}^{A^e}(A, A^\vee) & \xrightarrow{\cap} & \text{Tor}_0^{A^e}(A, A^\vee), \end{array}$$

where the injection $\lambda_1 : \text{HH}_{\text{sg}}^1(A, A) \rightarrow \text{Tor}_0^{A^e}(A, A^\vee)$ is defined in Proposition 6.2. Indeed, it is sufficient to prove that

$$f \cap (\alpha \otimes a_{1,n-1}) = \sum_i x_i f(a_{1,m}) \otimes y_i \in \text{Im}(\lambda_1) \quad (42)$$

for $f \in \text{HH}^m(A, A)$ and $\alpha \otimes a_{1,n-1} \in \text{Tor}_{n-1}^{A^e}(A, A^\vee)$. From the exact sequence in Proposition 6.2,

$$0 \longrightarrow \text{HH}_{\text{sg}}^{-1}(A, A) \xrightarrow{\lambda_1} A^\vee \otimes_{A^e} A \xrightarrow{\mu_1} \text{Hom}_{A^e}(A, A) \longrightarrow \text{HH}_{\text{sg}}^0(A, A) \rightarrow 0,$$

it follows that (42) is equivalent to

$$\mu_1(f \cap (\alpha \otimes a_{1,n-1})) = \sum_i x_i f(a_{1,m}) y_i = 0.$$

Indeed, we have

$$\begin{aligned} \sum_i x_i f(a_{1,m}) y_i &= \sum_i f(x_i a_1 \otimes a_{2,m}) y_i + \sum_i \sum_{j=1}^{m-1} (-1)^j f(x_i \otimes a_{1,j-1} \otimes a_j a_{j+1} \otimes a_{j+2,m}) y_i + \\ &\quad (-1)^m f(x_i \otimes a_{1,m-1}) a_m y_i \\ &= 0. \end{aligned}$$

Here we used $\delta(f) = 0$ in the first identity and $d(\alpha \otimes a_{1,n-1}) = 0$ in the second identity.

We are also interested in the case $n - m < 0$. Before discussing this case, let us define some operators on Hochschild cohomology and homology. Let A be an associative algebra (not necessarily, self-injective) over a field k . We will define a (generalized) cap product as follows. Let $0 \leq n < m$, define

$$\cap : C^m(A, A) \otimes C_n(A, A^\vee) \rightarrow C^{m-n-1}(A, A)$$

$$(f \cap (\alpha \otimes a_1 \otimes \cdots \otimes a_n))(b_1 \otimes \cdots \otimes b_{m-n-1}) := \sum_i x_i f(a_1 \otimes \cdots \otimes a_n \otimes y_i \otimes b_1 \otimes \cdots \otimes b_{m-n-1}),$$

where $\alpha(1) := \sum_i x_i \otimes y_i$.

Lemma 6.6. *The \cap induces a well-defined operator on the level of homology,*

$$\cap : \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_n(A, A^\vee) \rightarrow \mathrm{HH}^{m-n-1}(A, A).$$

Proof. It is sufficient to check that

$$\begin{aligned} Z^m(A, A) \cap Z_n(A, A^\vee) &\subset Z^{m-n-1}(A, A), \\ Z^m(A, A) \cap B_n(A, A^\vee) &\subset B^{m-n-1}(A, A), \\ B^m(A, A) \cap Z_n(A, A^\vee) &\subset B^{m-n-1}(A, A). \end{aligned}$$

Let $f \in Z^m(A, A)$ and $z = \sum \alpha \otimes a_1 \otimes \cdots \otimes a_n \in Z_n(A, A^\vee)$. Then we have

$$\begin{aligned} &\delta(f \cap z)(b_1 \otimes \cdots \otimes b_{m-n}) \\ &= b_1(f \cap z)(b_{2,m-n}) + \sum_{j=1}^{m-n-1} (-1)^j (f \cap z)(b_{1,j-1} \otimes b_j b_{j+1} \otimes b_{j+2,m-n}) + \\ &\quad (-1)^{m-n} (f \cap z)(b_{1,m-n-1}) b_{m-n} \\ &= b_1 \sum_k x_k f(a_{1,n} \otimes y_k \otimes b_{2,m-n}) + \sum_{j=1}^{m-n-1} (-1)^j \sum_k x_k f(a_{1,n} \otimes y_k \otimes b_{2,j-1} \otimes b_j b_{j+1} \otimes b_{j+2,m-n}) + \\ &\quad (-1)^{m-n} \sum_k x_k f(a_{1,n} \otimes y_k \otimes b_{1,m-n-1}) b_{m-n} \\ &= 0, \end{aligned}$$

where the last identity follows from the fact that $\delta(f) = 0$ and $dz = 0$. Similarly, for $f \in \mathrm{Hom}(A^{\otimes m-1}, A)$ and $z \in Z_n(A, A^\vee)$ we have

$$\delta(f) \cap z = \delta(f \cap z),$$

and for $f \in Z^m(A, A)$ and $z \in C_{n+1}(A, A^\vee)$

$$f \cap \partial_{n+1}(z) = \delta(f \cap z).$$

■

Let A be an associative algebra. We will also define a (generalized) cup product on $\mathrm{Tor}_*^{A^e}(A, A^\vee)$. Let $m, n \in \mathbb{Z}_{\geq 0}$,

$$\cup : C_m(A, A^\vee) \times C_n(A, A^\vee) \rightarrow C_{m+n+1}(A, A^\vee) \tag{43}$$

is defined as follows, take

$$\sum_i (x_i \otimes y_i) \otimes a_{1,m} \in C_m(A, A^\vee)$$

and

$$\sum_j (x'_j \otimes y'_j) \otimes b_{1,n} \in C_n(A, A^\vee),$$

we define

$$(\sum_i x_i \otimes y_i \otimes a_{1,m}) \cup (\sum_j x'_j \otimes y'_j \otimes b_{1,n}) := \sum_{i,j} (x_i \otimes y'_j y_i) \otimes a_{1,m} \otimes x'_j \otimes b_{1,n}.$$

Lemma 6.7. *The generalized cup product defined above is well-defined on $\text{Tor}_{\geq 0}^{A^e}(A, A^\vee)$. Moreover, it is graded commutative.*

Proof. Let

$$\alpha = \sum (x_i \otimes y_i) \otimes a_{1,m} \in Z_m(A, A^\vee)$$

and

$$\beta = \sum (x'_j \otimes y'_j) \otimes b_{1,n} \in Z_n(A, A^\vee),$$

then we have the following,

$$\begin{aligned} d(\alpha \cup \beta) &= d(\sum (x_i \otimes y'_j y_i) \otimes a_{1,m} \otimes x'_j \otimes b_{1,n}) \\ &= \sum (x_i a_1 \otimes y'_j y_i) \otimes a_{2,m} \otimes x'_j \otimes b_{1,n} + \\ &\quad \sum_{k=1}^{m-1} (-1)^k \sum (x_i \otimes y'_j y_i) \otimes a_{1,i-1} \otimes a_i a_{i+1} \otimes a_{i+2,m} \otimes x'_j \otimes b_{1,n} + \\ &\quad \sum (-1)^m \sum (x_i \otimes y'_j y_i) \otimes a_{1,m-1} \otimes a_m x'_j \otimes b_{1,n} + \\ &\quad \sum (-1)^{m+1} \sum (x_i \otimes y'_j y_i) \otimes a_{1,m} \otimes x'_j b_1 \otimes b_{2,n} + \\ &\quad \sum_{k=1}^{n-1} \sum (-1)^{m+1+k} (x_i \otimes y'_j y_i) \otimes a_{1,m} \otimes x'_j \otimes b_{1,k-1} \otimes b_k b_{k+1} \otimes b_{k+2,n} + \\ &\quad \sum (-1)^{m+n+1} (x_i \otimes b_n y'_j y_i) \otimes a_{1,m} \otimes x'_j \otimes b_{1,n-1} \\ &= 0, \end{aligned}$$

where the last identity follows from the fact that $d\alpha = d\beta = 0$. Hence we have

$$Z_m(A, A^\vee) \cup Z_n(A, A^\vee) \subset Z_{m+n+1}(A, A^\vee).$$

Similarly, the followings can be verified

$$\begin{aligned} Z_m(A, A^\vee) \cup B_n(A, A^\vee) &\subset B_{m+n+1}(A, A^\vee), \\ B_m(A, A^\vee) \cup Z_n(A, A^\vee) &\subset B_{m+n+1}(A, A^\vee). \end{aligned}$$

It remains to verify the graded commutativity. By calculation, we have

$$\alpha \cup \beta - (-1)^{mn} \beta \cup \alpha = \sum_{k=0}^m \sum_{j=0}^n (-1)^{m+n(k-1)} d((x_i \otimes y_i) \otimes a_{1,m-k} \otimes x'_j \otimes b_{1,n} \otimes y'_j \otimes a_{m-k+1,m}).$$

Hence, we have

$$\alpha \cup \beta - (-1)^{mn} \beta \cup \alpha \in B_{m+n+1}(A, A^\vee).$$

Thus, $\alpha \cup \beta = (-1)^{mn} \beta \cup \alpha$. Therefore, we have finished the proof. ■

Let us go back to our special case, then we have the following.

Lemma 6.8. *Let A be a self-injective algebra over a field k . Then the following diagram commutes for $m \geq 1, n \geq 2$ and $m - n \geq 1$,*

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^{m-n}(A, A) \\ \mathrm{id} \otimes \lambda_n \uparrow & & \cong \uparrow \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee) & \xrightarrow{\cap} & \mathrm{HH}^{m-n}(A, A). \end{array}$$

Proof. The proof is similar to the proof of Lemma 6.4. ■

Remark 6.9. For $m - n = 0$, we have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-m}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^0(A, A) \\ \mathrm{id} \otimes \lambda_n \uparrow & & \pi_0 \uparrow \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{m-1}^{A^e}(A, A^\vee) & \xrightarrow{\cap} & \mathrm{HH}^0(A, A), \end{array}$$

where $\pi_0 : \mathrm{HH}^0(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^0(A, A)$ is the surjection defined in Proposition 6.2.

Similarly, we also have the following lemma.

Lemma 6.10. *Let A be a self-injective algebra over a field k . Then we have the following commutative diagram for $m \geq 2, n \geq 2$,*

$$\begin{array}{ccc} \mathrm{HH}_{\mathrm{sg}}^{-m}(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^{-m-n}(A, A) \\ \lambda_m \otimes \lambda_n \uparrow & & \uparrow \lambda_{m+n} \\ \mathrm{Tor}_{m-1}^{A^e}(A, A^\vee) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A^\vee) & \xrightarrow{\cup} & \mathrm{Tor}_{m+n-1}^{A^e}(A, A^\vee). \end{array}$$

Remark 6.11. For $m = n = 1$, we have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) & \xrightarrow{\cup} & \mathrm{HH}_{\mathrm{sg}}^{-2}(A, A) \\ (\lambda_1 \otimes \lambda_1)^{-1} \downarrow & & \uparrow \lambda_{m+n} \\ \mathrm{Tor}_0^{A^e}(A, A^\vee) \otimes \mathrm{Tor}_0^{A^e}(A, A^\vee) & \xrightarrow{\cup} & \mathrm{Tor}_1^{A^e}(A, A^\vee). \end{array}$$

Therefore, in conclusion, the graded commutative associative algebra structure on $(\mathrm{HH}_{\mathrm{sg}}^*(A, A), \cup)$ becomes well-understood. Namely, it can be interpreted as the (generalized) cap product and the (generalized) cup product. Next we will investigate the graded Lie algebra structure on $(\mathrm{HH}_{\mathrm{sg}}^*(A, A), [\cdot, \cdot])$ in the case of a symmetric algebra A .

Before starting the case of symmetric algebras, let us recall the Connes B-operator on Hochschild homology and the structure of a Batalin-Vilkovisky (BV) algebra. For more details, we refer to [Con, Lod, Xu].

Definition 6.12. Let A be an associative algebra over a commutative algebra k . We define an operator on Hochschild homology,

$$B : C_r(A, A) \rightarrow C_{r+1}(A, A),$$

which sends $a_0 \otimes \cdots \otimes a_r \in C_r(A, A)$ to

$$B(a_0 \otimes \cdots \otimes a_r) := \sum_{i=0}^r (-1)^{ir} 1 \otimes a_i \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1} + (-1)^{ir} a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

It is easy to check that B is a chain map satisfying

$$B \circ B = 0,$$

which induces an operator (still denote by B),

$$B : \mathrm{HH}_r(A, A) \rightarrow \mathrm{HH}_{r+1}(A, A).$$

We call this operator Connes B -operator.

Remark 6.13. We also consider the Connes B -operator on normalized Hochschild complex.

$$\overline{B} : \overline{C}_r(A, A) \rightarrow \overline{C}_{r+1}(A, A),$$

which sends $a_{0,r} \in \overline{C}_r(A, A)$ to

$$\overline{B}(a_{0,r}) := \sum_{i=0}^r (-1)^{ir} 1 \otimes a_i \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

Definition 6.14. A Batalin-Vilkovisky algebra (BV algebra for short) is a Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\cdot, \cdot])$ together with an operator $\Delta : \mathcal{H}^* \rightarrow \mathcal{H}^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$, $\Delta(1) = 0$ and satisfying the following BV identity,

$$[\alpha, \beta] = (-1)^{|\alpha|+1} \Delta(\alpha \cup \beta) + (-1)^{|\alpha|} \Delta(\alpha) \cup \beta + \alpha \cup \Delta(\beta)$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}^*$.

From here onwards, assume that k is a field. Let A be a symmetric k -algebra (i.e. there is a symmetric, associative and non-degenerate inner product $\langle \cdot, \cdot \rangle : A \otimes A \rightarrow k$). Then the inner product $\langle \cdot, \cdot \rangle$ induces an A - A -bimodule isomorphism

$$\begin{aligned} t : A &\rightarrow D(A) := \mathrm{Hom}_k(A, k) \\ a &\mapsto \langle a, - \rangle. \end{aligned} \tag{44}$$

where the A - A -bimodule structure on $D(A)$ is given as follows, for $f \in D(A)$ and $a \otimes b \in A \otimes A^{\mathrm{op}}$,

$$((a \otimes b)f)(c) = f(cba).$$

This isomorphism t induces the following isomorphism

$$\begin{aligned} t \otimes \mathrm{id} : A \otimes A &\rightarrow D(A) \otimes A \cong \mathrm{End}(A) \\ a \otimes b &\mapsto t(a) \otimes b \mapsto (x \mapsto t(a)(x)b). \end{aligned}$$

We define the element

$$(t \otimes \mathrm{id})^{-1}(\mathrm{id}) := \sum_i e_i \otimes f_i \in A \otimes A$$

as the Casimir element of A (with respect to the inner product $\langle \cdot, \cdot \rangle$) (cf. [Brou]). The following proposition states some properties on Casimir element.

Proposition 6.15 (Proposition 3.3. [Brou]). 1. For all $a, a' \in A$, we have

$$\sum_i a e_i a' \otimes f_i = e_i \otimes a' f_i a.$$

2. The map

$$\begin{aligned} A &\rightarrow \text{Hom}_{A^e}(A, A^e) \\ a &\mapsto \sum_i e_i a \otimes f_i. \end{aligned}$$

is an right A - A -bimodule isomorphism. Here A is a right A - A -bimodule defined as follows, for $a \in A$ and $b \otimes c \in A^e$,

$$a \cdot (b \otimes c) := cab,$$

the right A - A -bimodule structure on $\text{Hom}_{A^e}(A, A^e)$ is given by, for $f \in \text{Hom}_{A^e}(A, A^e)$ and $b \otimes c \in A^e$,

$$f \cdot (b \otimes c)(a) := f(cba),$$

and we identify $\text{Hom}_{A^e}(A, A^e)$ as

$$(A \otimes A)^A := \left\{ \sum a_i \otimes b_i \in A \otimes A \mid \sum a a_i \otimes b_i = \sum a_i \otimes b_i a, \text{ for any } a \in A \right\}.$$

Since we have the following isomorphisms via the isomorphism t defined in (44) above,

$$\begin{aligned} D(C_n(A, A)) &\cong D(A \otimes_{A^e} D^2(A^{\otimes n+2})) \cong \text{Hom}_{A^e}(A, D(A^{\otimes n+2})) \\ &\cong \text{Hom}_{A^e}(D^2(A^{\otimes n+2}), D(A)) \cong C^n(A, A) \end{aligned}$$

where the third isomorphism follows from the fact that D induces an equivalence between A^e -mod and $(A^e\text{-mod})^{\text{op}}$ and the forth isomorphism is induced from the isomorphism t . Hence we have a duality between Hochschild homology and cohomology, for $n \in \mathbb{Z}_{\geq 0}$,

$$\text{HH}_n(A, A)^* \cong \text{HH}^n(A, A). \quad (45)$$

Hence from Proposition 6.2 and 6.15, we have for $n \geq 1$,

$$\kappa_n : \text{Ext}^1(A, \Omega^{n+2}(A)) \cong \text{HH}_{\text{sg}}^{-n-1}(A, A) \cong \text{Tor}_n^{A^e}(A, A^\vee) \cong \text{Tor}_n^{A^e}(A, A) \cong \text{HH}^n(A, A)^*.$$

In the rest of our paper, for simplicity, we often use the same κ_n to indicate any of those natural isomorphisms above. For example, we have the following isomorphism,

$$\begin{aligned} \kappa_n : \quad \text{Tor}_n^{A^e}(A, A) &\rightarrow \text{Ext}^1(A, \Omega^{n+2}(A)) \\ a_0 \otimes a_1 \otimes \cdots \otimes a_n &\mapsto (b \mapsto \sum_i d(e_i a_0 \otimes a_{1,n} \otimes f_i \otimes b \otimes 1)). \end{aligned}$$

Moreover we have the following result on symmetric algebras.

Theorem 6.16 ([Tra], [Men2]). Let A be a symmetric algebra over a field k . Then

$$(\text{HH}^*(A, A), \cup, [\cdot, \cdot], \Delta)$$

is a BV algebra, where the BV-operator Δ is the dual of the Connes B -operator via the duality (45) above.

Now let us state our propositions.

Proposition 6.17. *Let A be a symmetric algebra over a field k . Then we have the following commutative diagram for $m \geq 1, n \geq 2$ and $n - m \geq 1$,*

$$\begin{array}{ccccc}
& & \mathrm{HH}^m(A, A) \otimes \mathrm{Ext}_{A^e}^1(A, \Omega^{n+1}(A)) & \xrightarrow{[\cdot, \cdot]} & \mathrm{Ext}_{A^e}^m(A, \Omega^{n+1}(A)) \\
& \nearrow \mathrm{id} \otimes \kappa_n & \uparrow \cong & & \uparrow \cong \\
& & \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{[\cdot, \cdot]} & \mathrm{HH}_{\mathrm{sg}}^{m-n-1}(A, A) \\
& \searrow \mathrm{id} \otimes \kappa_n & \uparrow \mathrm{id} \otimes \kappa_n & & \uparrow \kappa_{n-m+1} \\
& & \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \mathrm{Tor}_{n-m}^{A^e}(A, A)
\end{array}$$

κ_{n-m+1} (curved arrow from $\mathrm{HH}_{\mathrm{sg}}^{m-n-1}(A, A)$ to $\mathrm{Tor}_{n-m}^{A^e}(A, A)$)

where $\{\cdot, \cdot\}$ is defined as follows, for $f \in \mathrm{HH}^m(A, A), \alpha \in \mathrm{Tor}_{n-1}^{A^e}(A, A)$,

$$\{f, \alpha\} := (-1)^m \Delta(f) \cap \alpha + f \cap B(\alpha) + (-1)^{m+1} B(f \cap \alpha).$$

Proof. From the definition of the Gerstenhaber bracket $[\cdot, \cdot]$, it follows that the top square is commutative. Hence it remains to check the commutativity of the outer square. Let $f \in \mathrm{HH}^m(A, A)$ and

$$z := \sum a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \in \mathrm{Tor}_{n-1}^{A^e}(A, A).$$

Then

$$\begin{aligned}
& \kappa_{n-m+1}(\{f, z\})(b_1 \otimes \cdots \otimes b_m) \\
&= \sum (-1)^{m+(m-1)(n-1)} d(e_j a_0 \Delta(f))(a_{1,m-1}) \otimes a_{m,n-1} \otimes f_j \otimes b_{1,m} \otimes 1 + \\
& \quad \sum_{i=0}^{n-1} \sum (-1)^{i(n-1)+mn} d(e_j(f \otimes \mathrm{id}))(a_{i,n-1} \otimes a_{0,i-1}) \otimes f_j \otimes b_{1,m} \otimes 1 + \\
& \quad \sum (-1)^{m+1+m(n-1)} d(e_j \otimes a_0 f)(a_{1,m}) \otimes a_{m+1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1 + \\
& \quad \sum_{i=1}^{n-m-1} \sum (-1)^{m+1+m(n-1)+i(n-m-1)} d(e_j \otimes a_{i+m,n-1} \otimes a_0 f)(a_{1,m}) \otimes a_{m+1,i+m-1} \otimes f_j \otimes b_{1,m} \otimes 1 \\
& [f, \kappa_n(z)](b_1 \otimes \cdots \otimes b_m) = f \bullet \kappa_n(z)(b_{1,m}) - \kappa_n(z) \circ f(b_{1,m}) \\
&= \sum_{i=1}^m \sum (-1)^{(m+1-i)(n+1)} d(f \otimes \mathrm{id})(b_{1,i-1} \otimes d(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_i \otimes 1) \otimes b_{i+1,m}) \otimes 1 + \\
& \quad \sum_{i=1}^{n+1} (-1)^{n+1+(n+1-i)m} d(\mathrm{id}_i \otimes f)(d(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_1 \otimes 1) \otimes b_{2,m} \otimes 1) - \\
& \quad \sum d(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes f(b_{1,m}) \otimes 1) \\
&= \sum_{i=1}^m \sum (-1)^{(m-i)(n+1)} d(f \otimes \mathrm{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 + \\
& \quad \sum_{i=1}^n \sum (-1)^{(n+1-i)m} d(\mathrm{id}_i \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1)
\end{aligned}$$

Claim that

$$\begin{aligned}
0 &= \sum_{i=1}^{n-m+1} \sum_{j=1}^{n-m+1} (-1)^{i(n-1)+mn} d(e_j(f \otimes \text{id}))(a_{i,n-1} \otimes a_{0,i-1}) \otimes f_j \otimes b_{1,m} \otimes 1) + \\
&\quad \sum_{j=1}^{n-m+1} (-1)^{m+1+m(n-1)} d(e_j \otimes a_0 f(a_{1,m}) \otimes a_{m+1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1) + \\
&\quad \sum_{i=1}^{n-m-1} \sum_{j=1}^{n-m-1} (-1)^{m+1+m(n-1)+i(n-m-1)} d(e_j \otimes a_{i+m,n-1} \otimes a_0 f(a_{1,m}) \otimes a_{m+1,i+m-1} \otimes f_j \otimes b_{1,m} \otimes 1) - \\
&\quad \sum_{i=1}^{n-m} \sum_{j=1}^{n-m} (-1)^{(n+1-i)m} d(\text{id}_i \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1)
\end{aligned}$$

Hence we have

$$\begin{aligned}
&(\kappa_{n-m+1}(\{f, z\}) - [f, \kappa_n(z)])(b_1 \otimes \cdots \otimes b_m) \\
&= \sum_{j=1}^{n-m+1} (-1)^{m+(m-1)(n-1)} d(e_j a_0 \Delta(f))(a_{1,m-1}) \otimes a_{m,n-1} \otimes f_j \otimes b_{1,m} \otimes 1) + \\
&\quad \sum_{j=1}^{n-m+1} (-1)^{mn} d(e_j f(a_{0,m-1}) \otimes a_{m,n-1} \otimes f_j \otimes b_{1,m} \otimes 1) + \\
&\quad \sum_{i=n-m+2}^{n-1} \sum_{j=1}^{n-1} (-1)^{i(n-1)+mn} d(e_j(f \otimes \text{id}))(a_{i,n-1} \otimes a_{0,i-1}) \otimes f_j \otimes b_{1,m} \otimes 1) - \\
&\quad \sum_{i=1}^m \sum_{j=1}^m (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 - \\
&\quad \sum_{i=n-m+1}^n \sum_{j=1}^n (-1)^{(n+1-i)m} d(\text{id}_i \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1)
\end{aligned} \tag{46}$$

We have the following

$$\begin{aligned}
&\sum_j d(\text{id}_n \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1) \\
&= \sum_{j,k} d(e_j a_0 \otimes a_{1,n-1} \otimes \langle e'_k f(f_j \otimes b_{1,m-1}), 1 \rangle f'_k \otimes b_m \otimes 1) \\
&= \sum_{j,k} \sum_{l=1}^{m-1} \sum_{i=l}^{m-1} (-1)^{(m+l)(m-i)+1} \delta(d(e_j a_0 \otimes a_{1,n-l} \otimes \langle f(\text{id}_{m-i-1} \otimes e'_k \otimes a_{n-l+1,n-1} \otimes f_j \otimes \text{id}_{i-l}), 1 \rangle f'_k \\
&\quad \otimes \text{id}_l \otimes 1)(b_{1,m}) + \sum_j \sum_{i=1}^m (-1)^{m(m-1)} d(e_j a_0 \otimes a_{1,n-1} \otimes \Delta(f)(b_{1,m-1}) f_j \otimes b_m \otimes 1) + \\
&\quad \sum_k \sum_{l=1}^m (-1)^{(m-1)(m+l)} d(f(b_{1,m-l} \otimes e'_k \otimes a_{n-l+1,n-1}) a_0 \otimes a_{1,n-l} \otimes f'_k \otimes b_{m-l+1,m} \otimes 1) + \\
&\quad \sum_j \sum_{l=1}^{m-1} (-1)^{ml+1} d(e_j a_0 \otimes a_{1,n-l-1} \otimes f(a_{n-l,n-1} \otimes f_j \otimes b_{1,m-l-1}) \otimes b_{m-l,m} \otimes 1
\end{aligned} \tag{47}$$

Combining (46) with (47), we obtain that

$$\begin{aligned}
& (\kappa_{n-m+1}(\{f, z\}) - [f, \kappa_n(z)])(b_1 \otimes \cdots \otimes b_m) \\
&= \sum (-1)^{mn} d(e_j f(a_{0,m-1}) \otimes a_{m,n-1} \otimes f_j \otimes b_{1,m} \otimes 1 + \\
& \quad \sum_{i=n-m+1}^{n-1} \sum (-1)^{i(n-1)+mn} d(e_j(f \otimes \text{id})(a_{i,n-1} \otimes a_{0,i-1}) \otimes f_j \otimes b_{1,m} \otimes 1) - \\
& \quad \sum_{i=1}^m \sum (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 - \\
& \quad \sum_j \sum_{l=1}^m (-1)^{(m-1)(m+l)+m} df(b_{1,m-l} \otimes e_j \otimes a_{n-l+1,n-1}) a_0 \otimes a_{1,n-l} \otimes f_j \otimes b_{m-l+1,m} \otimes 1
\end{aligned} \tag{48}$$

Let us compute the following term in (48).

$$\begin{aligned}
& \sum d(f \otimes \text{id})(b_{1,m-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_m) \otimes 1 \\
&= \sum df(b_{1,m-1} \otimes e_j a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 \\
&= (-1)^{m-1} \sum db_1 f(b_{2,m-1} \otimes e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 + \\
& \quad \sum_{i=1}^{m-2} \sum (-1)^{m-1+i} f(b_{1,i-1} \otimes b_i b_{i+1} \otimes b_{i+2,m-1} \otimes e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 + \\
& \quad f(b_{1,m-2} \otimes b_{m-1} e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 + df(b_{1,m-1} \otimes e_j) a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 \\
&= \sum_{i=1}^{m-1} (-1)^{m-1+(i-1)n} \sum \delta(df(\text{id}_{m-i-1} \otimes e_j \otimes a_{0,i-1}) \otimes a_{i,n-1} \otimes \text{id}_i \otimes 1)(b_{1,m}) \\
& \quad \sum_{i=1}^{m-1} (-1)^{(n-1)i+1} df(b_{1,m-1-i} \otimes e_j a_0 \otimes a_{1,i}) \otimes a_{i+1,n-1} \otimes f_j \otimes b_{m-i,m} \otimes 1 + \\
& \quad \sum_{i=1}^m (-1)^{(m-1)i+m+1} df(b_{1,m-i} \otimes e_j \otimes a_{n-i+1,n-1}) a_0 \otimes a_{1,n-i} \otimes f_j \otimes b_{m-i+1,m} \otimes 1 + \\
& \quad \sum (-1)^{mn} de_j f(a_{0,m-1}) \otimes a_{m,n-1} \otimes f_j \otimes b_{1,m} \otimes 1 + \\
& \quad \sum_{i=n-m+1}^{n-1} (-1)^{i(n-1)+mn} de_j(f \otimes \text{id})(a_{i,n-1} \otimes a_{0,i-1}) \otimes f_j \otimes b_{1,m} \otimes 1
\end{aligned} \tag{49}$$

From (49), it follows that

$$\kappa_{n-m+1}(\{f, z\}) - [f, \kappa_n(z)] \in Z^m(A, \Omega^{n+1}(A)),$$

hence we have the following identity in $\text{Ext}_{A^e}^m(A, \Omega^{n+1}(A))$

$$\kappa_{n-m+1}(\{f, z\}) = [f, \kappa_n(z)].$$

Therefore we have completed our proof. ■

Remark 6.18. We also have the following commutative diagram for $m \geq 1, n \geq 2$ and $n - m \geq 1$,

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{HH}^{n-1}(A, A)^* & \xrightarrow{[\cdot, \cdot]^*} & \mathrm{HH}^{n-m}(A, A)^* \\ \downarrow \mathrm{id} \otimes \kappa_n & & \downarrow \kappa_{m-n+1} \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \mathrm{Tor}_{n-m}^{A^e}(A, A) \end{array}$$

where $[\cdot, \cdot]^*$ is defined as follows, for any $f \in \mathrm{HH}^m(A, A)$ and $\alpha \in \mathrm{HH}^{n-1}(A, A)^*$,

$$[f, \alpha]^*(-) := \langle \alpha, [f, -] \rangle.$$

In fact, for $f \in \mathrm{HH}^m(A, A), \alpha \in \mathrm{Tor}_{n-1}^{A^e}(A, A)$ and $g \in \mathrm{HH}^{n-m}(A, A)$,

$$\begin{aligned} [f, \kappa_n^{-1}(\alpha)]^*(g) &= \langle \kappa_n^{-1}(\alpha), [f, g] \rangle \\ &= \langle \kappa_n^{-1}(\alpha), (-1)^m \Delta(f) \cup g + f \cup \Delta(g) + (-1)^{m+1} \Delta(f \cup g) \rangle \\ &= (-1)^m \langle \kappa_{m-n+1}^{-1}(\Delta(f) \cap \alpha), g \rangle + \langle \kappa_{m-n+1}^{-1}(B(f \cap \alpha)), g \rangle + \\ &\quad (-1)^{m+1} \langle \kappa_{m-n+1}^{-1}(f \cap B(\alpha)), g \rangle \\ &= \langle \kappa_{m-n+1}^{-1}(\{f, \alpha\}), g \rangle \end{aligned}$$

where the second identity is the BV identity (cf. Definition 6.14), hence it follows that the diagram above is commutative.

Similarly, we obtain the following proposition.

Proposition 6.19. *Let A be a symmetric algebra over a field k . Then we have the following commutative diagram for $m \geq 1, n \geq 2$ and $n - m \leq -2$,*

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{Ext}_{A^e}^1(A, \Omega^{n+1}(A)) & \longrightarrow & \mathrm{Ext}_{A^e}^m(A, \Omega^{n+1}(A)) \\ \uparrow & & \uparrow \\ \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-n}(A, A) & \xrightarrow{[\cdot, \cdot]} & \mathrm{HH}_{\mathrm{sg}}^{m-n-1}(A, A) \\ \mathrm{id} \otimes \kappa_n \uparrow & & \uparrow \kappa_{n-m+1} \\ \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{n-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \mathrm{HH}^{m-n-1}(A, A) \end{array}$$

where $\{\cdot, \cdot\}$ is defined as follows: for any $f \in \mathrm{HH}^m(A, A)$ and $\alpha \in \mathrm{Tor}_{n-1}^{A^e}(A, A)$,

$$\{f, \alpha\} := (-1)^m \Delta(f) \cap \alpha + f \cap B(\alpha) + (-1)^{m+1} \Delta(f \cap \alpha),$$

where α represents the generalized cap product defined in Lemma 6.6.

Proof. The proof is similar to the one of Proposition 6.17. Take $f \in \mathrm{HH}^m(A, A)$ and

$$z := \sum a_0 \otimes a_{1,n-1} \in \mathrm{Tor}_{n-1}^{A^e}(A, A).$$

Then we have

$$\begin{aligned}
& \kappa_{n-m+1}(\{f, z\})(b_{1,m}) \\
&= (-1)^m \sum de_j a_0 \Delta(f)(a_{1,n-1} \otimes f_j \otimes b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 + \\
& \quad \sum_{i=0}^{n-1} \sum (-1)^{i(n-1)} de_j f(a_{i,n-1} \otimes a_{0,i-1} \otimes f_j \otimes b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 + \\
& \quad (-1)^{m+1} d\Delta(f \cap z)(b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1
\end{aligned} \tag{50}$$

$$\begin{aligned}
& [f, \kappa_n(z)](b_1 \otimes \cdots \otimes b_m) \\
&= \sum_{i=1}^m \sum (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 + \\
& \quad \sum_{i=1}^n \sum (-1)^{(n+1-i)m} d(\text{id}_i \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1)
\end{aligned}$$

Let us compute the following term,

$$\begin{aligned}
& df(b_{1,m-1} \otimes e_j a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 \\
&= \sum (-1)^{m-1} \delta(df(\text{id}_{m-2} \otimes e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes \text{id} \otimes 1)(b_{1,m}) + \\
& \quad (-1) \sum df(b_{1,m-2} \otimes e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_{m-1} b_m \otimes 1 + \\
& \quad \sum df(b_{1,m-2} \otimes e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_{m-1} \otimes b_m + \\
& \quad \sum df(b_{1,m-2} \otimes b_{m-1} e_j \otimes a_0) \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 + \\
& \quad \sum df(b_{1,m-1} \otimes e_j) a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_m \otimes 1 \\
&= \sum_{i=1}^{m-1} \sum (-1)^{m-1+(i-1)n} \delta(df(\text{id}_{m-i-1} \otimes e_j \otimes a_{0,i-1}) \otimes a_{m-i-1,n-1} \otimes f_j \otimes \text{id}_{m-i} \otimes 1)(b_{1,m}) \\
& \quad \sum_{i=1}^n \sum (-1)^{(n-1)i+1} df(b_{1,m-i-1} \otimes e_j a_0 \otimes a_{1,i}) \otimes a_{i+1,n-1} \otimes f_j \otimes b_{m-i,m} \otimes 1 + \\
& \quad \sum_{i=1}^n \sum (-1)^{i+1} df(b_{1,m-i} \otimes e_j \otimes a_{n-i+1,n-1}) a_0 \otimes a_{1,n-i} \otimes f_j \otimes b_{m-i+1,m} \otimes 1 + \\
& \quad \sum_{i=1}^{n-1} \sum (-1)^{i(n-1)} e_j f(a_{i,n-1} \otimes a_{0,i-1} \otimes f_j \otimes b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1
\end{aligned} \tag{51}$$

Hence the following identity holds by combining (50) and (51).

$$\begin{aligned}
& (\kappa_{n-m+1}(\{f, z\}) - [f, \kappa_n(z)])(b_{1,m}) \\
&= (-1)^m \sum de_j a_0 \Delta(f)(a_{1,n-1} \otimes f_j \otimes b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 + \\
& \quad (-1)^{m+1} d\Delta(f \cap z)(b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 - \\
& \quad \sum_{i=1}^{m-n} \sum (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 - \\
& \quad \sum_{i=1}^n \sum (-1)^{(n+1-i)m} d(\text{id}_i \otimes f)(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m} \otimes 1) - \\
& \quad \sum_{i=1}^n \sum (-1)^{i+1} df(b_{1,m-i} \otimes e_j \otimes a_{n-i+1,n-1}) a_0 \otimes a_{1,n-i} \otimes f_j \otimes b_{m-i+1,m} \otimes 1
\end{aligned} \tag{52}$$

By calculation, we have the following identity

$$\begin{aligned}
& \sum de_j a_0 \otimes a_{1,n-1} \otimes f(f_j \otimes b_{1,m-1}) \otimes b_m 1 \\
&= \sum de_j a_0 \otimes a_{1,n-1} \otimes \langle e'_k f(f_j \otimes b_{1,m-1}), 1 \rangle f'_k \otimes b_m \otimes 1 \\
&= \sum_{l=1}^n \sum_{i=l}^{m-1} (-1)^{(m+l)(m-i)+1} \delta(de_j a_0 \otimes a_{1,n-l} \otimes \langle f(\text{id}_{m-i-1} \otimes e'_k \otimes a_{n-l+1,n-1} \otimes f_j \otimes \text{id}_{i-l}), 1 \rangle f'_k \\
& \quad \otimes \text{id}_l \otimes 1)(b_{1,m}) + \sum_{i=1}^m \sum (-1)^{m(m-1)} de_j a_0 \otimes a_{1,n-1} \otimes \Delta(f)(b_{1,m-1}) f_j \otimes b_m \otimes 1 + \\
& \quad \sum_{l=1}^{n-1} \sum (-1)^{ml+1} de_j a_0 \otimes a_{1,n-l-1} \otimes f(a_{n-l,n-1} \otimes f_j \otimes b_{1,m-l-1}) \otimes b_{m-l,m} \otimes 1 + \\
& \quad \sum_{i=0}^{n-1} \sum (-1)^{(m+1)i+1} de_j a_0 \otimes a_{1,n-i-1} \otimes \langle f(b_{1,m-i-1} \otimes e'_j \otimes a_{n-i,n-1}) f_j, 1 \rangle f'_j \otimes b_{m-i,m} \otimes 1 + \\
& \quad \sum_{i=1}^{m-n} \sum de_j a_0 \langle f(b_{i,m-n-1} \otimes e'_j \otimes a_{1,n-1} \otimes f_j \otimes b_{1,i-1}), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1
\end{aligned} \tag{53}$$

Combining (52) and (53), we obtain that

$$\begin{aligned}
& (\kappa_{n-m+1}(\{f, z\}) - [f, \kappa_n(z)])(b_{1,m}) \\
&= (-1)^{m+1} \Delta(f \cap z)(b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 - \\
& \quad \sum_{i=1}^{m-n} \sum (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 - \\
& \quad \sum_{i=1}^{m-n} \sum de_j a_0 \langle f(b_{i,m-n-1} \otimes e'_j \otimes a_{1,n-1} \otimes f_j \otimes b_{1,i-1}), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1
\end{aligned} \tag{54}$$

Hence from (54), it remains to verify the following identity,

$$\begin{aligned}
0 = & (-1)^{m+1} \Delta(f \cap z)(b_{1,m-n-1}) \otimes b_{m-n,m} \otimes 1 - \\
& \sum_{i=1}^{m-n} \sum_{j=1}^{m-n} (-1)^{(m-i)(n+1)} d(f \otimes \text{id})(b_{1,i-1} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m}) \otimes 1 - \\
& \sum_{i=1}^{m-n} \sum_{j=1}^{m-n} d e_j a_0 \langle f(b_{i,m-n-1} \otimes e'_j \otimes a_{1,n-1} \otimes f_j \otimes b_{1,i-1}), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1
\end{aligned} \tag{55}$$

Let us prove Identity (55) above.

$$\begin{aligned}
& \sum d \langle e_j a_0 f(a_{1,n-1} \otimes f_j \otimes b_{1,m-n-1} \otimes e'_j), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 \\
= & \sum (-1)^{n-1} d \langle f(e_j a_0 \otimes a_{1,n-1} \otimes f_j b_1 \otimes b_{2,m-n-1} \otimes e'_j), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 + \\
& \sum_{i=1}^{m-n-2} (-1)^{n-1+i} \sum d \langle f(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,i-1} \otimes b_i b_{i+1} \otimes b_{i+2,m-n-1} \otimes e'_j), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 + \\
& \sum (-1)^m d \langle f(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m-n-2} \otimes b_{m-n-1} e'_j), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 + \\
& \sum (-1)^{m+1} d \langle f(e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{1,m-n-1}) e'_j, 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 \\
= & \sum_{i=1}^{m-n} \sum_{j=1}^{m-n} (-1)^{m+i-1} d f(b_{1,i} \otimes e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m-n-1}) \otimes b_{m-n,m} \otimes 1 + \\
& \sum_{i=1}^{m-n-1} \sum_{j=1}^{m-n-1} (-1)^{i(m-n)+1} \langle e_j a_0 f(a_{1,n-1} \otimes f_j \otimes b_{i,m-n-1} \otimes e'_j \otimes b_{1,i-1}), 1 \rangle f'_j \otimes b_{m-n,m} \otimes 1 + \\
& \sum_{i=1}^{m-n} \sum_{j=1}^{m-n} d \langle f(b_{1,i-1} \otimes e'_k e_j a_0 \otimes a_{1,n-1} \otimes f_j \otimes b_{i,m-n-1}), 1 \rangle f'_k \otimes b_{m-n,m} \otimes 1.
\end{aligned}$$

Hence we have the right hand side in (55) is zero. ■

Proposition 6.20. *Let A be a symmetric algebra over a field k . Then we have the following commutative diagram for $m \geq 2, n \geq 2$,*

$$\begin{array}{ccc}
\text{Ext}^1(A, \Omega^{m+1}(A)) \otimes \text{Ext}_{A^e}^1(A, \Omega^{n+1}(A)) & \xrightarrow{[\cdot, \cdot]} & \text{Ext}_{A^e}^1(A, \Omega^{m+n+2}(A)) \\
\uparrow & & \uparrow \\
\text{HH}_{\text{sg}}^{-m}(A, A) \otimes \text{HH}_{\text{sg}}^{-n}(A, A) & \xrightarrow{[\cdot, \cdot]} & \text{HH}_{\text{sg}}^{-m-n-1}(A, A) \\
\uparrow \kappa_m \otimes \kappa_n & & \uparrow \kappa_{m+n+1} \\
\text{Tor}_{m-1}^{A^e}(A, A) \otimes \text{Tor}_{n-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \text{Tor}_{m+n}^{A^e}(A, A)
\end{array}$$

where $\{\cdot, \cdot\}$ is defined as follows, for any $\alpha \in \text{Tor}_{m-1}^{A^e}(A, A)$ and $\beta \in \text{Tor}_{m-1}^{A^e}(A, A)$

$$\{\alpha, \beta\} := (-1)^m B(\alpha) \cup \beta + \alpha \cup B(\beta) + (-1)^{m+1} B(\alpha \cup \beta),$$

where \cup represents the generalized cup product defined in (43).

Proof. The proof is similar to the proof in Proposition 6.19. ■

Therefore, combining Theorem 6.16, Proposition 6.17, 6.19 and 6.20, we obtain the following corollary.

Corollary 6.21. *Let A be a symmetric algebra over a field k . Then $\mathrm{HH}_{\mathrm{sg}}^*(A, A)$ is a BV algebra with BV operator Δ_{sg} , which is the Connes B -operator for the negative part $\mathrm{HH}_{\mathrm{sg}}^{<0}(A, A)$, the Δ -operator for the positive part $\mathrm{HH}_{\mathrm{sg}}^{>0}(A, A)$ and*

$$\Delta_{\mathrm{sg}}|_{\mathrm{HH}_{\mathrm{sg}}^0(A, A)} = 0 : \mathrm{HH}_{\mathrm{sg}}^0(A, A) \rightarrow \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A).$$

In particular, we have two BV subalgebras $\mathrm{HH}_{\mathrm{sg}}^{\leq 0}(A, A)$ and $\mathrm{HH}_{\mathrm{sg}}^{\geq 0}(A, A)$ with induced BV algebra structures.

Proof. It remains to prove that we have the following commutative diagram for $m \in \mathbb{Z}_{>0}$, that is, the image of the bracket $\{\cdot, \cdot\}$ is contained in $\mathrm{HH}_{\mathrm{sg}}^{-1}(A, A)$.

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{m-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \mathrm{Tor}_0^{A^e}(A, A) \\ & \searrow & \uparrow \\ & & \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) \end{array} \quad (56)$$

where we recall that the injection

$$\mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) \rightarrow \mathrm{Tor}_0^{A^e}(A, A)$$

is defined in Proposition 6.2 and

$$\{f, \alpha\} := (-1)^m \Delta(f) \cap \alpha + f \cap B(\alpha)$$

for any $f \in \mathrm{HH}^{m-1}(A, A)$ and $\alpha \in \mathrm{Tor}_{m-1}^{A^e}(A, A)$. From the short exact sequence in Proposition 6.2,

$$0 \longrightarrow \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) \longrightarrow A^\vee \otimes_{A^e} A \xrightarrow{\mu^*} \mathrm{Hom}_{A^e}(A, A) \longrightarrow \mathrm{HH}_{\mathrm{sg}}^0(A, A) \rightarrow 0$$

it is sufficient to show that for any $f \in \mathrm{HH}^{m-1}(A, A)$ and

$$\alpha := \sum a_0 \otimes a_{1, m-1} \in \mathrm{Tor}_{m-1}^{A^e}(A, A),$$

we have

$$\mu^*(\{f, \alpha\}) = 0.$$

Indeed, we have

$$\begin{aligned} \mu^*(\{f, \alpha\}) &= \sum_j (-1)^m e_j a_0 \Delta(f)(a_{1, m-1}) f_j + \sum_j \sum_{i=0}^{m-1} (-1)^{i(m-1)} e_j f(a_{i, m-1} \otimes a_{0, i-1}) f_j \\ &= \sum_j (-1)^m e_j a_0 \langle \Delta(f)(a_{1, m-1}) f_j e'_k, 1 \rangle f'_k + \sum_j \sum_{i=0}^{m-1} (-1)^{i(m-1)} e_j f(a_{i, m-1} \otimes a_{0, i-1}) f_j \\ &= \sum_j \sum_{i=1}^m (-1)^{m+i(m-1)} e_j a_0 \langle f(a_{i, m-1} \otimes f_j e'_k \otimes a_{1, i-1}), 1 \rangle f'_k + \\ &\quad \sum_j \sum_{i=0}^{m-1} (-1)^{i(m-1)} e_j f(a_{i, m-1} \otimes a_{0, i-1}) f_j \\ &= 0 \end{aligned}$$

since by direct calculation, we obtain that

$$\sum_j \sum_{i=1}^m (-1)^{m+i(m-1)} e_j a_0 \langle f(a_{i,m-1} \otimes f_j e'_k \otimes a_{1,i-1}), 1 \rangle f'_k = 0,$$

$$\sum_j \sum_{i=0}^{m-1} (-1)^{i(m-1)} e_j f(a_{i,m-1} \otimes a_{0,i-1}) f_j = 0.$$

Moreover, we have the following commutative diagram,

$$\begin{array}{ccc} \mathrm{HH}^m(A, A) \otimes \mathrm{Tor}_{m-1}^{A^e}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A) \\ \downarrow \mathrm{id} \otimes \kappa_m & & \parallel \\ \mathrm{HH}^m(A, A) \otimes \mathrm{HH}_{\mathrm{sg}}^{-m}(A, A) & \xrightarrow{[\cdot, \cdot]} & \mathrm{HH}_{\mathrm{sg}}^{-1}(A, A). \end{array}$$

Therefore, the proof has been completed. ■

Corollary 6.22. *Let A be a symmetric algebra over a field k . Then the cyclic homology $\mathrm{HC}_*(A, A)$ is a graded Lie algebra of lower degree 2, that is, $\mathrm{HC}_*(A, A)[-1]$ is a graded Lie algebra.*

Proof. This is an immediate corollary of Proposition 26 in [Men1] since from Corollary 6.21 above it follows that $\mathrm{HH}_*(A, A)$, equipped with Connes B-operator is a BV algebra. ■

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